Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning: Introduction

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1 Motivation

Quantitative vs. qualitative

Spatio-temporal configurations can be described quantitatively by specifying the coordinates of the relevant objects:

> **Example**: *At time point 10.0 object A is at position* $(11.0, 1.0, 23.7)$*, at time point 11.0 at position* $(15.2, 3.5, 23.7)$*. From time point 0.0 to 11.0, object B is at position* $(15.2, 3.5, 23.7)$*. Object C is at time point 11.0 at position* $(300.9, 25.6, 200.0)$ *and at time point 35.0 at* $(11.0, 1.0, 23.7)$*.*

Quantitative vs. qualitative

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

> **Example**: *Object A hit object B. Afterwards, object C arrived.*

Sometimes we want to reason with such descriptions, e.g.:

> *Object C was not close to object A when it hit object B.*

Representation of qualitative knowledge

Intention: Description of configurations using a finite vocabulary and reasoning about these descriptions

- Specification of a vocabulary: usually a finite set of relations (often binary) that are pairwise disjoint and exhaustive
- Specification of a language: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal semantics
- Analysis of computational properties and design of reasoning methods (often constraint propagation)
- Perhaps, specification of operational semantics for verifying whether a relation holds in a given quantitative configuration

Applications in ...

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- ...

Qualitative temporal relations: *Point Calculus*

We want to talk about time instants (points) and binary relations over them.

- **Vocabulary**:
  - X equals Y: $X = Y$
  - X before Y: $X < Y$
  - X after Y: $X > Y$
- **Language**:
  - Allow for disjunctions of basic relations to express indefinite information. Use set of relations to express that. For instance, $\{<,=\}$ expresses $\leq$.
  - 2^3 different relations (including the **impossible** and the **universal** relation)
  - Use sets of atomic formulae with these relations to describe configurations. For example: $\{x\{=\}y, y\{<,>\}z\}$
- **Semantics**: Interpret the time point symbols and relation symbols over the rational (or real) numbers.

Some reasoning problems

$$\left\{x\{<,=\}y, y\{<,=\}z, v\{<,=\}y, w\{>\}y, z\{<,=\}x\right\}$$

- Satisfiability: Are there values for all time points such that all formulae are satisfied?
- Satisfiability with $v\{=\}w$?
- Finding a satisfying instantiation of all time points
- Deduction: Does $x\{=\}y$ logically follow?
  Does $v\{<,=\}w$ follow?
- Finding a minimal description: What are the most constrained relations that describe the same set of instantiations?

2 Constraint Satisfaction Problems

From a logical point of view …

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in satisfiability, model finding, and deduction
- **Constraint Satisfaction Problems**

CSP – Definition

Definition

A constraint satisfaction problem (CSP) is given by

- a set V of n variables $\{v_1, \ldots, v_n\}$,
- for each v_i, a value domain D_i
- constraints (relations over subsets of the variables)

Tasks:

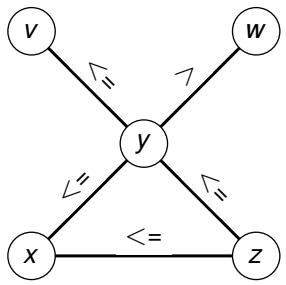
Find one (or all) solution(s), i. e., tuples

$$(d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n$$

such that the assignment $v_i \mapsto d_i$ $(1 \leq i \leq n)$ satisfies all constraints.

Our example: Point relations

- Our point relation CSP is a binary CSP with infinite domains.
- It can be represented as a **constraint graph**:



CSP – Example

k-colorability: Can we color the nodes of a graph with k colors in a way such that all nodes connected by an edge have different colors?

- The node set is the set of variables
- The domain of each variable is $\{1, \ldots, k\}$
- The constraints are that nodes connected by an edge must have a different value

Note: This CSP has a particular restricted form:

- Only binary constraints
- The domains are finite

Other examples: Many problems (e.g. cross-word puzzle, n-queens problem, configuration, ...) can be cast as a CSP (and solved this way)

Computational complexity

Theorem

It is NP-hard to decide solvability of CSPs, even binary CSPs.

Proof.

Since k-colorability is NP-complete (even for fixed $k \geq 3$), solvability of CSPs in general must be NP-hard. □

Question: Is CSP solvability in NP?

3 Constraint Solving Methods

Solving CSP

- Enumeration of all assignments and testing

⇝ ... too costly

- Backtracking search

⇝ 1001 different strategies, often "dead" search paths are explored extensively

- Constraint propagation: elimination of obviously impossible values followed by backtracking search Interleaving backtracking and constraint propagation
- Many other search methods, e.g., local search, stochastic search, etc.

⇝ How do we solve CSP with infinite domains?

General assumptions

- Only at most binary constraints (i.e., we can use constraint graph)
- Uniform domain D for all variables
- Unary constraints D_i and binary constraints R_{ij} are sets of values or sets of pairs of values, resp.
- We assume that for all nodes i, j:

$$(x, y) \in R_{ij} \Rightarrow (y, x) \in R_{ji}$$

Local consistency

- A CSP is **locally consistent** if for particular subsets of the variables, solutions of the restricted CSP can be extended to solutions of a larger set of variables.

⇝ Methods to transform a CSP into a tighter, but "equivalent" problem.

Definition

A binary CSP $\langle V, D, C \rangle$ is arc-consistent (or 2-consistent) if for all nodes $1 \leq i, j \leq n$,

$$x \in D_i \Rightarrow \exists y \in D_j \text{ s. t. } (x, y) \in R_{ij}$$

⇝ When a CSP is arc-consistent, each one variable assignment $\{v_i\} \to D$ that satisfies all (unary) constraints in v_i, i. e., D_i can be extended to a two variable assignment $\{v_i, v_j\} \to D$ that satisfies all unary/binary constraints in these variables, i. e., D_i, D_j, and R_{ij}.

Arc consistency

EnforceArcConsistency ($\mathcal{C}$):
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but arc-consistent CSP $\mathcal{C}'$

repeat
  for each arc (v_i, v_j) with $R_{ij} \in C$
    $D_i := D_i \cap \{x \in D : \text{ex. } y \in D_j \text{ s. t. } (x, y) \in R_{ij}\}$
  endfor
until no domain is changed

- Terminates in time $O(n^3 \cdot k^3)$ if we have finite domains (where k is the number of values)

⇝ There exist different (more efficient) algorithms for enforcing arc consistency.

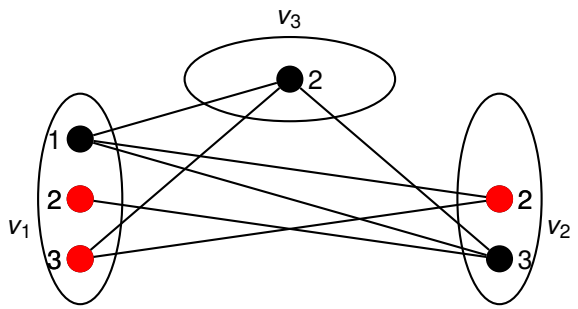
Arc consistency

Lemma

- *Enforcing arc consistency yields an arc consistent CSP.*
- *Enforcing arc consistency is solution invariant, i. e. it does not change the set of solutions.*

⇝ Arc-consistent CSPs need not be consistent, and vice versa.

Arc consistency – Example



$$\begin{aligned} D_1 &= \{1,2,3\} \\ D_2 &= \{2,3\} \\ D_3 &= \{2\} \\ R_{ij} &= \text{"}\neq\text{"} \text{ for } i \neq j \end{aligned}$$

- Since all unary constraints are singletons, this is a solution of the CSP.

this is a unique solution of the original CSP.

Local consistency (2): Path consistency

Definition

A binary CSP $\langle V, D, C \rangle$ is said to be path-consistent (or 3-consistent) if for all nodes $1 \leq i, j, k \leq n$,

$$x \in D_i, y \in D_j, (x,y) \in R_{ij} \Rightarrow \\ \exists z \in D_k \text{ s.t. } (x,z) \in R_{ik} \text{ and } (y,z) \in R_{jk}$$

⇝ When a CSP is path-consistent, each two variable assignment $\{v_i, v_j\} \to D$ satisfying all constraints in v_i and v_j can be extended to any three variable assignment $\{v_i, v_j, v_k\} \to D$ such that all constraints in these variables are satisfied.

Path consistency

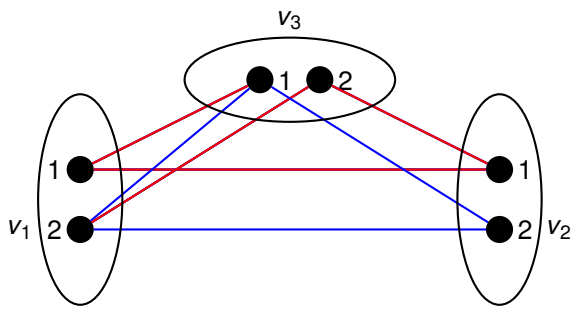
EnforcePathConsistency $(\mathcal{C})$:
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$ of size n
Output: an equivalent, but path-consistent CSP $\mathcal{C}'$

repeat
  for all $1 \leq i, j, k \leq n$
    $R_{ij} := R_{ij} \cap$
      $\{(x,y) : \text{ex. } z \in D_k \text{ s.t. } (x,z) \in R_{ik} \text{ and } (y,z) \in R_{jk}\}$
  endfor
until no binary constraint is changed

⇝ Terminates in time $O(n^5 \cdot k^5)$ if we have finite domains (where k is the number of values)

⇝ Enforcing path consistency is solution invariant.

Path consistency – Example



Local consistency (2): Path consistency

Definition

A binary CSP $\langle V, D, C \rangle$ is said to be path-consistent (or 3-consistent) if for all nodes $1 \leq i, j, k \leq n$,

$$x \in D_i, y \in D_j, (x, y) \in R_{ij} \Rightarrow \exists z \in D_k \text{ s.t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}$$

$\rightsquigarrow$ When a CSP is path-consistent, each two variable assignment $\{v_i, v_j\} \to D$ satisfying all constraints in v_i and v_j can be extended to any three variable assignment $\{v_i, v_j, v_k\} \to D$ such that all constraints in these variables are satisfied.

Local consistency (3): k-consistency and strong k-consistency

Definition

- A binary CSP $\langle V, D, C \rangle$ is k-consistent if, given variables $x_1, \ldots, x_k$ and an assignment $a : \{x_1, \ldots, x_{k-1}\} \to D$ that satisfies all constraint in these variables, a can be extended to an assignment $a' : \{x_1, \ldots, x_k\} \to D$ that satisfies all constraints in these k variables.
- A binary CSP $\langle V, D, C \rangle$ is strongly k-consistent if it is k'-consistent for each $k' \leq k$.
- A binary CSP $\langle V, D, C \rangle$ is globally consistent if it is strongly n-consistent where n is the size of V.

Local consistency (3)

- k-consistency: The computation costs grow exponentially with k.
- If a CSP is globally consistent, then
  - a solution can be constructed in polynomial time,
  - its constraints are minimal,
  - and it has a solution iff there is no empty constraint.
- k-consistent $\not\Rightarrow$ $k-1$-consistent

4 Qualitative Constraint Satisfaction Problems

Qualitative reasoning with CSP

If we want to use CSPs for qualitative reasoning, we have

- infinite domains
- mostly only finitely many relations (basic relations and their unions)
- arc-consistent CSPs (usually)

Questions:

- How do we achieve k-consistency (for some fixed k)?
- Is k-consistency (for some fixed k) enough to guarantee global consistency?

Operations on binary relations

Composition:

$$R_1 \circ R_2 = \{(x,y) \in D^2 : \exists z \in D \text{ s.t. } (x,z) \in R_1 \text{ and } (z,y) \in R_2\}$$

Converse:

$$R^{-1} = \{(x,y) \in D^2 : (y,x) \in R\}$$

Intersection:

$$R_1 \cap R_2 = \{(x,y) \in D^2 : (x,y) \in R_1 \text{ and } (x,y) \in R_2\}$$

Union:

$$R_1 \cup R_2 = \{(x,y) \in D^2 : (x,y) \in R_1 \text{ or } (x,y) \in R_2\}$$

Complement:

$$\overline{R} = \{(x,y) \in D^2 : (x,y) \notin R\}$$

Conditions on vocabulary for qualitative reasoning

- Let **B** be a finite set of (binary) base relations.
- ⇝ The relations in **B** should be JEPD, i. e., jointly exhaustive and pairwise disjoint.
- **B** should be **closed under converse**.
- Let **A** be the set of relations that can be built by taking the unions of relations from **B** (⇝ $2^{|\mathbf{B}|}$ different relations).
- ⇝ **A** is closed under converse, complement, intersection, union.
- **A** should be **closed under composition of base relations**, i. e., for all $B, B' \in \mathbf{B}$, $B \circ B' \in A$.
- ⇝ **A** is closed under composition of arbitrary relations.
- ⇝ This condition does not hold necessarily.
  Example: $\mathbf{B} = \{<, =, >\}$ interpreted over the integers is not closed under composition (and has no finite closure):

$$< \circ < = < \setminus \{(i,j) : i = j - 1\} \subsetneq <$$

Computing operations on relations

Let **A** be a relation system over the set of base relations **B** that satisfies the conditions spelled out above.

- ⇝ We may write relations as sets of base relations:

$$B_1 \cup \cdots \cup B_n \sim \{B_1, \ldots, B_n\}$$

Then the operations on the relations can be computed as follows:

Composition:

$$\{B_1, \ldots B_n\} \circ \{B'_1, \ldots, B'_m\} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (B_i \circ B'_j)$$

Converse:

$$\{B_1, \ldots, B_n\}^{-1} = \{B_1^{-1}, \ldots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \ldots, B_n\}} = \{B \in \mathbf{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and union are defined set-theoretically.

Reasoning problems

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition xRy: Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y the strongest constrained (minimal) relation entailed by the CSP.

$\rightsquigarrow$ These problems are equivalent under Turing reductions

Reductions between CSP problems

Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.

CSAT $\leq_T$ CENT and CENT $\leq_T$ CMIN are obvious.

CENT $\leq_T$ CSAT: We solve CENT ($CSP \models xRy$?) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let $B_1, \ldots, B_k$ be the relations for which we get a positive answer. Then $x\{B_1, \ldots, B_k\}y$ is entailed by the CSP.

CMIN $\leq_T$ CENT: We use entailment for computing the minimal constraint for each pair. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. $\square$

Path consistency for qualitative CSPs

Given a qualitative CSP with $R_{ij} = R_{ji}^{-1}$. Then path consistency can be enforced by doing the following:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}).$$

Path consistency guarantees ...

- sometimes minimality
- sometimes satisfiability
- however sometimes the CSP is not satisfiable, even if the CSP contains only base relations

$\rightsquigarrow$ All this depends on the vocabulary.

Example: Point relations

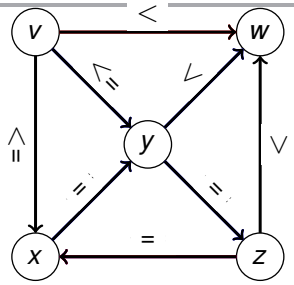
Composition table:

	$<$	$=$	$>$
$<$	$<$	$<$	$<,=,>$
$=$	$<$	$=$	$>$
$>$	$<,=,>$	$>$	$>$

Figure: Composition table for the point algebra. For example: $\{<\} \circ \{=\} = \{<\}$

- $\{<,=\} \circ \{<\} = \{<\}$
- $\{<,>\} \circ \{<\} = \{<,=,>\}$
- $\{<,=\}^{-1} = \{>,=\}$
- $\{<,=\} \cap \{>,=\} = \{=\}$

Qualitative reasoning with CSP – Example



$$
\begin{aligned}
R_{vw} &= R_{vw} \cap R_{vy} \circ R_{yw} \\
&= R_{vw} \cap ((< \circ <) \cup (= \circ <)) \\
&= \{<,=,>\} \cap (\{<\} \cup \{<\}) \\
&= \{<\}
\end{aligned}
$$

$$
\begin{aligned}
R_{vx} &= R_{vx} \cap R_{vy} \circ R_{yx} \\
&= R_{vx} \cap ((< \circ =) \cup (< \circ >) \cup (= \circ =) \cup (= \circ >)) \\
&= \{<,=,>\}
\end{aligned}
$$

Some properties of the point relations

Theorem

A path-consistent CSP over the point relations is consistent.

Corollary

CSAT, CENT and CMIN are polynomial problems for the point relations.

Theorem

A path-consistent CSP over all point relations without $\{<,>\}$ is minimal.

Proofs later ...

5 A Pathological Relation System

A pathological relation system

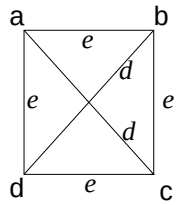
Let e, d, i be (self-converse) base relations between points on a circle:

- e: Rotation by 72 degrees (left or right)
- d: Rotation by 144 degrees (left or right)
- i: Identity

Composition table:

$$
\begin{aligned}
e \circ e &= \{i, d\} \\
d \circ d &= \{i, e\} \\
e \circ d &= \{e, d\} \\
d \circ e &= \{e, d\}
\end{aligned}
$$

The following CSP is path-consistent and contains only base relations, but it is not satisfiable:



6 Outlook

Outlook

- Qualitative representation and reasoning usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- Reasoning problems are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with CSP methods (but note: infinite domains).
- Path consistency is the basic reasoning step . . . sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.
- Can be taken further $\rightsquigarrow$ relation algebra

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A Pathological Relation System
Outlook
Literature

7 Literature

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Literature I

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In S. C. Shapiro, editor, Encyclopedia of Artificial Intelligence, pages 205–211. Wiley, Chichester, England, 1987.

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Motivation
CSP
Solving CSP
Qualitative CSP
A Pathological Relation System
Outlook
Literature

Literature II

R. Hirsch.
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