Principles of Knowledge Representation and Reasoning

Answer Set Programming

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1 Introduction

ASP: Background

- Answer set semantics: a formalization of negation as failure in logic programming (Prolog)
- Several formal semantics: well-founded semantics, perfect-model semantics, inflationary semantics, ...
- Can be viewed as a simpler variant of default logic

ASP: Negation as failure

- Another interpretation for negation: not\(x\) \(\equiv\) "It cannot be shown that\(x\) is true"
- For example, you are innocent until proven guilty

Example

\[]{}\textit{innocent} \leftarrow \textit{not guilty}.
ASP: Declarative problem solving

- What is the problem? instead of: How to solve the problem?
- Outsourcing the computation part to an external solver

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\textbf{Problem} \quad \textbf{Solution}
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\textbf{Modeling} \quad \textbf{Interpretation}
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\textbf{Representation} \quad \textbf{Computation} \quad \textbf{Output}
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### 2 Answer Sets

- Normal logic programs
- Interpretation and Satisfiability
- Definition
- Formal properties
- Stratification

### Normal logic programs I

Let $\mathcal{A}$ be a set of first-order atoms.

**Rules:**

$$a \leftarrow b_1, \ldots, b_m, \neg c_1, \ldots, \neg c_k$$

where $\{a, b_1, \ldots, b_m, c_1, \ldots, c_k\} \subseteq \mathcal{A}$

- Meaning similar to default logic:
  1. we have derived $b_1, \ldots, b_m$ and
  2. cannot derive any of $c_1, \ldots, c_k$,
  then derive $a$.
- Rules without right-hand side (facts): $a \leftarrow$
- Rules without left-hand side (constraints):
  $$\leftarrow b_1, \ldots, b_m, \neg c_1, \ldots, \neg c_k$$

### Normal logic programs II

Let $\mathcal{A}$ be a set of first-order atoms.

**Rules:**

$$a \leftarrow b_1, \ldots, b_m, \neg c_1, \ldots, \neg c_k$$

where $\{a, b_1, \ldots, b_m, c_1, \ldots, c_k\} \subseteq \mathcal{A}$

- $a$ is called the head of the rule, denoted by $\text{head}(r)$.
- The literals $b_1, \ldots, b_m$ form the positive body of $r$, denoted by $\text{body}^+(r)$.
- The literals $\neg c_1, \ldots, \neg c_k$ form the negative body of $r$, denoted by $\text{body}^-(r)$.
- The body of $r$ is the union of positive and negative body:
  $$\text{body}(r) = \text{body}^+(r) \cup \text{body}^-(r).$$
Normal logic programs: Example

Example

\[
\begin{align*}
\text{bird}(X) & \leftarrow \text{eagle}(X) \\
\text{bird}(X) & \leftarrow \text{penguin}(X) \\
\text{fly}(X) & \leftarrow \text{bird}(X), \neg \text{nonfly}(X) \\
\text{nonfly}(X) & \leftarrow \text{penguin}(X) \\
\text{eagle}(\text{eddy}) & \leftarrow \\
\text{penguin}(\text{tweety}) & \leftarrow
\end{align*}
\]

Herbrand base and grounded rules

Example

\[
\begin{align*}
\text{bird}(\text{eddy}) & \leftarrow \text{eagle}(\text{eddy}) \\
\text{bird}(\text{tweety}) & \leftarrow \text{eagle}(\text{tweety}) \\
\text{bird}(\text{eddy}) & \leftarrow \text{penguin}(\text{eddy}) \\
\text{bird}(\text{tweety}) & \leftarrow \text{penguin}(\text{tweety}) \\
\text{fly}(\text{eddy}) & \leftarrow \text{bird}(\text{eddy}), \neg \text{nonfly}(\text{eddy}) \\
\text{fly}(\text{tweety}) & \leftarrow \text{bird}(\text{tweety}), \neg \text{nonfly}(\text{tweety}) \\
\text{nonfly}(\text{eddy}) & \leftarrow \text{penguin}(\text{eddy}) \\
\text{nonfly}(\text{tweety}) & \leftarrow \text{penguin}(\text{tweety}) \\
\text{eagle}(\text{eddy}) & \leftarrow \\
\text{penguin}(\text{tweety}) & \leftarrow
\end{align*}
\]

Satisfaction

A Herbrand interpretation is a subset \( X \) of the Herbrand base.

Satisfaction relation:

- \( X \models a \) if \( a \in X \).
- \( X \models r \) if \( \{b_1, \ldots, b_m\} \subseteq X \) or \( \{a, c_1, \ldots, c_n\} \cap X \neq \emptyset \),
  where \( r = a \leftarrow b_1, \ldots, b_m, \neg c_1, \ldots, \neg c_k \).
- \( X \models P \) if \( X \models r \) for each \( r \in P \).

Idea

Idea: “models” as interpretations that are satisfying, stable, and supported.
Positive *(not-free)* logic programs

**Definition (Answer set)**

Let $P$ be a logic program **without** not, $X \subseteq \text{atoms}(P)$. $X$ is the (unique) answer set of $P$ if it is the least fixpoint of the operator:

$$\Gamma_P(X) = \{a : \exists r = a \leftarrow b_1, \ldots, b_m \in P \text{ with } \{b_1, \ldots, b_m\} \subseteq X\}.$$

**Example**

$$P = \{a \leftarrow b, \quad d \leftarrow f, \quad b \leftarrow\}.$$

$$\Gamma^0 = \emptyset, \quad \Gamma^1 = \Gamma(\emptyset) = \{b\}, \quad \Gamma^2 = \Gamma(\Gamma^1) = \{b, d, a\}, \quad \Gamma^3 = \Gamma(\Gamma^2) = \{b, d, a, c\}, \quad \Gamma^4 = \Gamma(\Gamma^3) = \{b, d, a, c\} = \Gamma^3.$$

**Proposition**

If an atom $a$ belongs to an answer set of a logic program $P$, then $a$ is the head of one of the rules of $P$.

**Gelfond-Lifschitz reduct**

**Definition (Reduct)**

The reduct of a program $P$ with respect to a set of atoms $X \subseteq \text{atoms}(P)$ is defined as:

$$P^X := \{\text{head}(r) \leftarrow \text{body}^*(r) : r \in P, \quad c \notin X \text{ for each not } c \in \text{body}^-(r)\}.$$

That is, given $X$,

- ... delete all rules whose negative part contradicts $X$
- ... remove all negated atoms from the remaining rules

**Definition (Answer set)**

$X \subseteq \text{atoms}(P)$ is an answer set of $P$ if $X$ is an answer set of $P^X$.

**Some properties I**

**Proposition**

If an atom $a$ belongs to an answer set of a logic program $P$, then $a$ is the head of one of the rules of $P$.

**Proposition**

Each answer set of a normal logic program $P$ is a minimal model of $P$, i.e., it satisfies all rules in $P$ and there is no proper subset of $P$ satisfying all rules in $P$.

**Notice:** The converse is not true: not each minimal model is an answer set.
Proposition

Let $F$ be a set of (non-constraint) rules and $G$ be a set of constraints. A set of atoms $X$ is an answer set of $F \cup G$ iff it is an answer set of $F$ that satisfies $G$.

Proof.

$F \subseteq F \cup G$ implies $F^X \subseteq (F \cup G)^X$ and hence $\text{Ifp}_T(F^X) \subseteq \text{Ifp}_T((F \cup G)^X)$.

$\Rightarrow$: Assume $X$ is an answer set of $F \cup G$, hence $X = \text{Ifp}_T(F \cup G)^X$ and $X \models G$. Since $G$ contains constraints only, it follows that each $a \in X$ is the head of some rule in $F$. Hence, $X \subseteq \text{Ifp}_T(F^X)$, and thus $X$ is an answer set of $F$ that satisfies $G$.

$\Leftarrow$: Similar.

Difference to Propositional Logic

- The ancestor relation is the transitive closure of the parent relation.
- Transitive closure cannot be (concisely) represented in propositional/predicate logic.

\[
\text{par}(X, Y) \rightarrow \text{anc}(X, Y) \\
\text{par}(X, Z) \land \text{anc}(Z, Y) \rightarrow \text{anc}(X, Y)
\]

The above formulae only guarantee that \text{anc} is a superset of the transitive closure of \text{par}.

- For transitive closure one needs the minimality condition in some form: nonmonotonic logics, fixpoint logics, ...

Complexity: Existence of answer sets is NP-complete

- Membership in NP: Guess $X \subseteq \text{atoms}(P)$ (nondet. polytime), compute $P^X$, compute its closure, compare to $X$ (everything det. polytime).
- NP-hardness: Reduction from 3SAT: an answer set exists iff the following clauses are satisfiable:

\[
p \leftarrow \neg \hat{p} \land \hat{p} \leftarrow \neg p.
\]

for every propositional variable $p$ occurring in the clauses, and

\[
\neg \hat{l}_1, \neg \hat{l}_2, \neg \hat{l}_3
\]

for every clause $l_1 \lor l_2 \lor l_3$, where $l_i' = p$ if $l_i = \neg p$ and $l_i' = \hat{p}$ if $l_i = \neg \neg p$.

Stratification

The reason for multiple answer sets is the fact that $a$ may depend on $b$ and simultaneously $b$ may depend on $a$. The lack of this kind of circular dependencies makes reasoning easier.

Definition

A logic program $P$ is stratified if $P$ can be partitioned to $P = P_1 \cup \cdots \cup P_n$ so that for all $i \in \{1, \ldots, n\}$ and $(a \leftarrow b_1, \ldots, b_m, \neg c_1, \ldots, \neg c_k) \in P_i$,

- there is no $\neg a$ in $P_i$ and
- there are no occurrences of $a$ anywhere in $P_1 \cup \cdots \cup P_{i-1}$.

-
### Stratification

**Theorem**

A stratified program $P$ has exactly one answer set. The unique answer set can be computed in polynomial time.

**Example**

Our earlier examples with more than one or no answer sets:

\[
P_3 = \{ p \leftarrow \neg p \}
\]

\[
P_4 = \{ p \leftarrow \neg q, \ q \leftarrow \neg p \}
\]

### Programs for Reasoning with Answer Sets

- smodels (Niemelä & Simons), dlv (Eiter et al.), clasp (Schaub et al.), ...
- Schematic input:

  
  \[
  \begin{align*}
  p(X) & :\neg \text{not } q(X). \\
  q(X) & :\neg \text{not } p(X). \\
  r(a). & \\
  r(b). & \\
  r(c). & \\
  \text{anc}(X,Y) & : \text{par}(X,Y).
  \end{align*}
  \]

### AnsProlog

- Propositions are any combination of lowercase letters.
- Variables are any combination of letters starting with an uppercase letter.
- Write "\(\leftarrow\)" instead of $\leftarrow$.
- Integers can be used and so can arithmetic operations (+, −, *, /, %).
- Negation as failure is denoted by not.
- Strong negation is denoted by $\neg$.
- #const n = ... statements can be used to define constants.
- The #hide/#show statements can be used to influence which iterals are shown in the solution.
AnsProlog: Choice functions

- The literal \{b_1; \ldots; b_m\}
  is true iff any subset of the set \{b_1, \ldots, b_m\} is true.

Example
Generate all interpretations over the atoms \(a(1), a(2), a(3)\):
\{ a(1); a(2); a(3) \}.
With strong negation:
\(-a(X) :- not a(X), X=1..3. \ \{ a(1..3) \}.

AnsProlog: Choice with cardinality

- The literal \(l \{b_1; \ldots; b_m\} u\)
  is true iff at least \(l\) and at most \(u\) atoms (included) are true
  within the set \{b_1, \ldots, b_m\}.

Example
Generate all interpretations over the atoms \(a(1), a(2), a(3), b(1), b(2)\)
that contain exactly 2 true atoms:
\(2 \{ a(1..3); b(1..2) \} 2.\)
Generate all interpretations over the atoms \(a(1), a(2), a(3), b(1), b(2),
b(3)\) that do not contain exactly 2 or more true atoms for the same
predicate:
\(\{ a(1..3); b(1..3) \}.\)
\(:- 2 \{ a(1..3) \} 3.\)
\(:- 2 \{ b(1..3) \} 3.\)

AnsProlog: Domains of variables

- The domain of a variable must be known in order to avoid
  “unsafe”-error while the program is grounded.
- The domain can be set literal-wise, rule-wise, or program
  wise.
- For limiting the scope within a literal use the syntax:
  \(a(X) : \text{dom}(X)\)
  or \(a(X) : X=1..3\)

Example

\(\text{num}(0..10).\)
\(\text{even}(2*X) :- \text{num}(X), 2*X \leq 10.\)
\(1 \{ a(X) : \text{even}(X) \} 1.\)
\#show a/1.

Example: Graph coloring

\#const n = 2.
c(1..n).
\(1 \{ \text{color}(X,I) : c(I) \} 1 :- v(X).\)
\(:- \text{color}(X,I), \text{color}(Y,I), e(X,Y), c(I).\)

% Instance
\(v(1..4).\)
\(e(1,2).\)
\(e(1,3).\)
\(e(2,4).\)
\(e(3,4).\)
\% e(2,3).
\#show color/2.
Generate and test

ASP programs are often organized in a “generate-and-test” style: first describe candidate solutions, then rule out possible solutions by stating constraints.

Example

% n-Queens encoding %
#const n = 4.

% Generate possible positions %
1 { q(I,1..n) } 1 :- I = 1..n.

% Rule out attacking positions %
:- q(I1,J), q(I2,J), I1 != I2.
:- q(I,J), q(I+D,J+D), D = 1..n.
:- q(I,J), q(I+D,J-D), D = 1..n.

Generate and test: Further example

Problem: In a graph find cliques of size $\geq n$

Example

#const n = 3.

edge(X,Y) :- edge(Y,X).

n {clique(X) : node(X)}.

:- clique(X), clique(Y), node(X), node(Y), X#/=Y, not edge(X,Y).

% Instance %

node(1..5).

edge(1,2;4).

edge(2,3;4).

edge(3,4).

edge(4,2;5).

#show clique/1.

AnsProlog: Miscellaneous

The language is even bigger than that! It includes
- Disjunction in the head
- Other operators: #sum,#min,#max,#even,#odd,#avg, ...
- Multi-criteria optimizations
- Heuristic optimizations
- ...

(More on that in the exercises!)

Literature

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Francois Fages.
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