Principles of Knowledge Representation and Reasoning
Semantic Networks and Description Logics IV: Description Logics – Algorithms

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1 Motivation
Reasoning problems & algorithms

Reasoning problems:
- Satisfiability or subsumption of concept descriptions
- Satisfiability or instance relation in ABoxes

Solving techniques presented in this chapter:
- Structural subsumption algorithms
  - Normalization of concept descriptions and structural comparison
  - Very fast, but can only be used for small DLs
- Tableau algorithms
  - Similar to modal tableau methods
  - Often the method of choice
2 Structural Subsumption Algorithms

- Idea
- Example
- Algorithm
- Soundness
- Completeness
- Generalizations
- ABox Reasoning
In what follows we consider the rather small logic $\mathcal{FL}^-$:

- $C \sqcap D$
- $\forall r. C$
- $\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

   $$\forall r. C \sqcap \forall r. D \rightarrow \forall r. (C \sqcap D).$$

2. Compare all conjuncts with each other. For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.
Example

Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}.\text{Human} \sqcap \forall \text{has-child}.\exists \text{has-child} \]

\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}.(\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]

Check: \( C \subseteq D \)

1. Collect value restrictions in \( D \):
   \[
   \ldots \forall \text{has-child}.(\text{Human} \sqcap \exists \text{has-child})
   \]

2. Compare:
   1. For \( \text{Human} \) in \( D \), we have \( \text{Human} \) in \( C \).
   2. For \( \exists \text{has-child} \) in \( D \), we have \( \exists \text{has-child} \) in \( C \).
   3. For \( \forall \text{has-child}.(\ldots) \) in \( D \), we have \( \text{Human} \) and \( \exists \text{has-child} \) in \( C \).

\( \rightsquigarrow C \) is subsumed by \( D \)!
### Subsumption algorithm

**SUB(C, D) algorithm:**

1. Reorder terms (using commutativity, associativity and value restriction law):

   \[
   C = \bigcap A_i \cap \bigcap \exists r_j \cap \bigcap \forall r_k : C_k
   \]
   \[
   D = \bigcap B_i \cap \bigcap \exists s_m \cap \bigcap \forall s_n : D_n
   \]

2. For each \( B_i \) in \( D \), is there an \( A_i \) in \( C \) with \( A_i = B_i \)?

3. For each \( \exists s_m \) in \( D \), is there an \( \exists r_j \) in \( C \) with \( s_m = r_j \)?

4. For each \( \forall s_n : D_n \) in \( D \), is there a \( \forall r_k : C_k \) in \( C \) such that \( s_n = r_k \) and \( C_k \subseteq D_n \) (i.e., check \( \text{SUB}(C_k, D_n) \))?  

\( \Rightarrow C \sqsubseteq D \) iff all questions are answered positively.
Soundness

Theorem (Soundness)

\( \text{SUB}(C, D) \Rightarrow C \subseteq D \)

Proof sketch.

Reordering of terms step (1):

1. Commutativity and associativity are trivial

2. Value restriction law. We show: \((\forall r. (C \cap D))^I = (\forall r. C \cap \forall r. D)^I\)

   Assume \(d \in (\forall r. (C \cap D))^I\).

   If there is no \(e \in D\) with \((d, e) \in r^I\) it follows trivially that \(d \in (\forall r. C \cap \forall r. D)^I\).

   If there is an \(e \in D\) with \((d, e) \in r^I\) it follows \(e \in (C \cap D)^I = C^I \cap D^I\).

   Since \(e\) is arbitrary, we have \(d \in (\forall r. C)^I\) and \(d \in (\forall r. D)^I\), i.e., \((\forall r. (C \cap D))^I \subseteq (\forall r. C \cap \forall r. D)^I\).

   The other direction is similar.

Steps (2+3+4): Induction on the nesting depth of \(\forall\)-expressions.
Completeness

Theorem (Completeness)

\[ C \sqsubseteq D \Rightarrow SUB(C, D). \]

Proof idea.

One shows the contrapositive:

\[ \neg SUB(C, D) \Rightarrow C \not\sqsubseteq D \]

Idea: If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element \( d \) such that

\[ d \in C^I, \text{ but } d \notin D^I. \]
Generalizing the algorithm

Extensions of $\mathcal{FL}^-$ by

- $\neg A$ (atomic negation),
- $(\leq nr), (\geq nr)$ (cardinality restrictions),
- $r \circ s$ (role composition)

do not lead to any problems.

**However:** If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.

**More precisely:** There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

**Reason:** Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).
ABox reasoning

\textit{Idea: Abstraction + classification}

- \textbf{Complete} ABox by propagating value restrictions to role fillers.
- Compute for each object its \textit{most specialized concepts}.
- These can then be handled using the ordinary subsumption algorithm.
Motivation

Structural Subsumption Algorithms

Tableau Subsumption Method

Example
Reductions: Unfolding & Unsatisfiability
Model Construction
Equivalences & NNF
Constraint Systems
Transforming Constraint Systems
Invariances
Soundness and Completeness
Space Complexity
ABox Reasoning
Tableau method

Logic $\mathcal{ALC}$:

- $C \sqcap D$
- $C \sqcup D$
- $\neg C$
- $\forall r.C$
- $\exists r.C$

_Idea:_ Decide (un-)satisfiability of a concept description $C$ by trying to systemically construct a model for $C$. If that is successful, $C$ is satisfiable. Otherwise, $C$ is unsatisfiable.
Example: Subsumption in a TBox

Example

TBox:

\[
\begin{align*}
\text{Hermaphrodite} & = \text{Male} \sqcap \text{Female} \\
\text{Parent-of-sons-and-daughters} & = \\
& \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \\
\text{Parent-of-hermaphrodite} & = \exists \text{has-child}. \text{Hermaphrodite}
\end{align*}
\]

Query:

\[
\begin{align*}
\text{Parent-of-sons-and-daughters} & \subseteq \tau \\
\text{Parent-of-hermaphrodites}
\end{align*}
\]
Reductions

1. **Unfolding:**
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqsubseteq \exists \text{has-child}.(\text{Male} \sqcap \text{Female}) \]

2. **Reduction to unsatisfiability:** Is the concept
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \neg \exists \text{has-child}.(\text{Male} \sqcap \text{Female}) \]
   unsatisfiable?

3. **Negation normal form** (move negations inside):
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]

4. **Try to construct a model**
Model construction (1)

1 Assumption: There exists an object $x$ in the interpretation of our concept:

$$x \in (\exists \ldots)^{\mathcal{I}}$$

2 This implies that $x$ is in the interpretation of all conjuncts:

$$x \in (\exists \text{has-child}. \text{Male})^{\mathcal{I}}$$
$$x \in (\exists \text{has-child}. \text{Female})^{\mathcal{I}}$$
$$x \in (\forall \text{has-child}. (\neg \text{Male} \sqcup \neg \text{Female}))^{\mathcal{I}}$$

3 This implies that there should be objects $y$ and $z$ such that $(x,y) \in \text{has-child}^{\mathcal{I}}, (x,z) \in \text{has-child}^{\mathcal{I}}, y \in \text{Male}^{\mathcal{I}}$ and $z \in \text{Female}^{\mathcal{I}}$, and...
Model construction (2)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]
Model construction (3)

\[
x: \exists \text{has-child}. \text{Male} \\
x: \exists \text{has-child}. \text{Female} \\
x: \forall \text{has-child}. (\neg \text{Male} \sqcup \neg \text{Female})
\]

\[
\begin{array}{c}
x \\
\downarrow \text{has-child} \\
y \\
\quad \text{Male} \\
\quad (\neg \text{Male} \lor \neg \text{Female})
\end{array}
\quad \begin{array}{c}
x \\
\downarrow \text{has-child} \\
z \\
\quad \text{Female} \\
\quad (\neg \text{Male} \lor \neg \text{Female})
\end{array}
\]
Model construction (4)

\[
\begin{align*}
  x & : \exists \text{has-child}. \text{Male} \\
  x & : \exists \text{has-child}. \text{Female} \\
  x & : \forall \text{has-child}. (\neg \text{Male} \sqcap \neg \text{Female}) \\
  y & : \neg \text{Male}
\end{align*}
\]
Model construction (5)

\[ x : \exists \text{has-child}. \text{Male} \]
\[ x : \exists \text{has-child}. \text{Female} \]
\[ x : \forall \text{has-child}. (\neg \text{Male} \lor \neg \text{Female}) \]
\[ y : \neg \text{Female} \]
\[ z : \neg \text{Male} \]

\[ \models \]

Model constructed!
Tableau method (1): NNF

We write: $C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$. Now we have the following equivalences:

\[
\neg(C \cap D) \equiv \neg C \cup \neg D \quad \neg(C \cup D) \equiv \neg C \cap \neg D
\]
\[
\neg(\forall r.C) \equiv \exists r.\neg C \quad \neg(\exists r.C) \equiv \forall r.\neg C
\]
\[
\neg\neg C \equiv C
\]

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).

**Theorem (NNF)**

*The negation normal form of an $\mathcal{ALC}$ concept can be computed in polynomial time.*
Tableau method (2): Constraint systems

A constraint is a syntactical object of the form:

\[ x : C \quad \text{or} \quad xry, \]

where \( C \) is a concept description in NNF, \( r \) is a role name, and \( x \) and \( y \) are variable names.

Let \( \mathcal{I} \) be an interpretation with universe \( \mathcal{D} \). An \( \mathcal{I} \)-assignment \( \alpha \) is a function that maps each variable symbol to an object of the universe \( \mathcal{D} \).

A constraint \( x : C (xry) \) is satisfied by an \( \mathcal{I} \)-assignment \( \alpha \) if \( \alpha(x) \in C^\mathcal{I} \) (resp. \((\alpha(x), \alpha(y)) \in r^\mathcal{I}\)).
Tableau method (3): Constraint systems

Definition

A **constraint system** \( S \) is a finite, non-empty set of constraints. An \( \mathcal{I} \)-assignment \( \alpha \) **satisfies** \( S \) if \( \alpha \) satisfies each constraint in \( S \). \( S \) is **satisfiable** if there exist \( \mathcal{I} \) and \( \alpha \) such that \( \alpha \) satisfies \( S \).

Theorem

An \( \mathcal{ALC} \) concept \( C \) in NNF is satisfiable if and only if the system \( \{ x : C \} \) is satisfiable.
Tableau method (4): Transforming constraint systems

Transformation rules:

1. \( S \rightarrow \cap \{x : C_1, x : C_2\} \cup S \)
   if \((x : C_1 \cap C_2) \in S\) and either \((x : C_1) \in S\) or \((x : C_2) \in S\) or both are not in \(S\).

2. \( S \rightarrow \sqcup \{x : D\} \cup S \)
   if \((x : C_1 \sqcup C_2) \in S\) and neither \((x : C_1) \in S\) nor \((x : C_2) \in S\)
   and \(D = C_1\) or \(D = C_2\).

3. \( S \rightarrow \exists \{x r y, y : C\} \cup S \)
   if \((x : \exists r. C) \in S\), \(y\) is a fresh variable, and there is no \(z\) s.t.
   \((x r z) \in S\) and \((z : C) \in S\).

4. \( S \rightarrow \forall \{y : C\} \cup S \)
   if \((x : \forall r. C), (x r y) \in S\) and \((y : C) \not\in S\).

Notice: Deterministic rules (1,3,4) vs. non-deterministic (2).

Generating rules (3) vs. non-generating (1,2,4).
Tableau method (5): Invariances

Theorem (Invariance)

Let $S$ and $T$ be constraint systems.

1. If $T$ has been generated by applying a deterministic rule to $S$, then $S$ is satisfiable if and only if $T$ is satisfiable.

2. If $T$ has been generated by applying a non-deterministic rule to $S$, then $S$ is satisfiable if $T$ is satisfiable. Furthermore, if a non-deterministic rule can be applied to $S$, then it can be applied such that $S$ is satisfiable if and only if the resulting system $T$ is satisfiable.

Theorem (Termination)

Let $C$ be an $\mathcal{ALC}$ concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{ x : C \}$. 
Tableau method (6): Soundness and completeness

A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form \( x : A \) and \( x : \neg A \), where \( A \) is a concept name.

**Theorem (Soundness and Completeness)**

* A closed constraint system is satisfiable if and only it does not contain a clash.

**Proof idea.**

\( \Rightarrow \): obvious. \( \Leftarrow \): Construct a model by using the concept labels.
Space requirements

Because the tableau method is non-deterministic ($\rightarrow \sqcap$ rule), there could be exponentially many closed constraint systems in the end.

Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

Example

$$\exists r.A \sqcap \exists r.B \sqcap \forall r.( \exists r.A \sqcap \exists r.B \sqcap \forall r.(\ldots)))$$

However: One can modify the algorithm so that it needs only polynomial space.

Idea: Generate a $y$ only for one $\exists r.C$ and then proceed into the depth.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):

- **Normalize** and **unfold** and add inequalities for all pairs of objects mentioned in the ABox.
- **Strictly speaking**, in $\mathcal{ALC}$ we do not need this because we are never forced to identify two objects.
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