Principles of Knowledge Representation and Reasoning
Semantic Networks and Description Logics I: Simple, Strict Inheritance Networks

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November 16, 2015
Introduction
Terminological reasoning

- Often, we need to use semantic (conceptual, terminological) knowledge . . .

- For example, consider a knowledge base that classifies things into different categories, which in turn may be organized in some hierarchical way

  Task: Query objects that belong to a specific category or one of its super categories . . .

- Even more involved: Answer queries of users of the knowledge base who are not aware of the internal categories of the knowledge base

- Topic of this section: a naïve (maybe too naïve) approach to reasoning with terminological knowledge, namely inheritance networks
Definition

A strict inheritance network is defined by a set of nodes (representing concepts, properties) and a set of directed edges (representing generalization, the is-a-relation).
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A strict inheritance network is defined by a set of nodes (representing concepts, properties) and a set of directed edges (representing generalization, the is-a-relation).

- **Reasoning problem**: Is some concept $C$ a specialization (a subconcept) of another concept $C'$?
- ... and how can we solve this problem efficiently?
A simple network formalism
Networks as formula sets

A strict inheritance network can be seen as a set $\Theta$ of formulae of the form

$$C_1 \text{ isa } C_2.$$
Networks as formula sets

A strict inheritance network can be seen as a set $\Theta$ of formulae of the form

$$C_1 \textbf{isa} C_2.$$

Example

- Student $\textbf{isa}$ Person
- Student $\textbf{isa}$ studious
- Professor $\textbf{isa}$ Person
- Professor $\textbf{isa}$ knowledgeable
- Grad-Student $\textbf{isa}$ Student
- Undergrad-Student $\textbf{isa}$ Student
Networks as formula sets

A strict inheritance network can be seen as a set $\Theta$ of formulae of the form

$$C_1 \text{ isa } C_2.$$ 

Example

Student isa Person
Student isa studious
Professor isa Person
Professor isa knowledgeable
Grad-Student isa Student
Undergrad-Student isa Student

Reasoning problem (inheritance problem): $\Theta \models C_1 \text{ isa } C_2$?
Logical semantics

We assign the following logical semantics to $\text{isa}$-formulae:

$$C_1 \text{ isa } C_2 \iff \forall x. C_1(x) \rightarrow C_2(x)$$
Logical semantics

- We assign the following logical semantics to isa-formulae:

\[ C_1 \text{ isa } C_2 \iff \forall x. C_1(x) \rightarrow C_2(x) \]

- ...i.e., we interpret each directed edge or isa-formula as a universally quantified implication.
Logical semantics

- We assign the following logical semantics to \textit{isa}-formulae:

\[ C_1 \text{ isa } C_2 \iff \forall x. C_1(x) \rightarrow C_2(x) \]

- ...i.e., we interpret each directed edge or \textit{isa}-formula as a universally quantified implication.

- This is intuitively plausible: each instance of a sub-concept is an instance of the super-concept.
Logical semantics

- We assign the following logical semantics to `isa`-formulae:

  \[ C_1 \text{ isa } C_2 \iff \forall x. C_1(x) \rightarrow C_2(x) \]

- ...i.e., we interpret each directed edge or `isa`-formula as a universally quantified implication.

- This is intuitively plausible: each instance of a sub-concept is an instance of the super-concept.

- Now we can reduce the inheritance problem as follows: Let \( \pi(\Theta) \) be the translation. Then we want to know:

  \[ \pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x) ? \]
Logical semantics

- We assign the following logical semantics to \texttt{isa}-formulae:
  
  \[ C_1 \texttt{isa} C_2 \iff \forall x. C_1(x) \rightarrow C_2(x) \]

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- Now we can reduce the inheritance problem as follows: Let \( \pi(\Theta) \) be the translation. Then we want to know:

  \[ \pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x) ? \]

- How hard is this problem?
A polynomial reasoning algorithm

Let $G_\Theta$ be the graph corresponding to $\Theta$. Then we have:

$$\pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x)$$

iff

there exists a path in $G_\Theta$ from $C_1$ to $C_2$. 
A polynomial reasoning algorithm

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... which has to be proven (next slides).
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- Thus, we have reduced reasoning in strict inheritance networks to graph reachability problem, which is solvable in polynomial time.
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Thus, we have reduced reasoning in strict inheritance networks to graph reachability problem, which is solvable in polynomial time.

Note: Reasoning is not simple because we used a graph to represent the knowledge (there are actually very difficult graph problems),
A polynomial reasoning algorithm

Let \( G_\Theta \) be the graph corresponding to \( \Theta \). Then we have:

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\pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x)
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- … which has to be proven (next slides).
- Thus, we have reduced reasoning in strict inheritance networks to graph reachability problem, which is solvable in polynomial time.
- **Note**: Reasoning is not simple because we used a graph to represent the knowledge (there are actually very difficult graph problems),
- … reasoning is simple because the expressiveness compared with first-order logic is very restricted.
Soundness

Theorem (Soundness of inheritance reasoning)

If there exists a path from $C_1$ to $C_2$ in $G_\Theta$, then

$$\pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x).$$
Soundness

Theorem (Soundness of inheritance reasoning)

If there exists a path from $C_1$ to $C_2$ in $G_\Theta$, then

$$\pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x).$$

Proof.

If there is a path, then there exists a chain of implications of the form

$$\forall x. D_j(x) \rightarrow D_{j+1}(x)$$

with $D_0 = C_1$ and $D_n = C_2$.

Since logical implication is transitive, the claim follows trivially.
Theorem (Completeness of inheritance reasoning)

If \( \pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x) \), then there exists a path from \( C_1 \) to \( C_2 \) in \( G_\Theta \).
Completeness

Theorem (Completeness of inheritance reasoning)

\[ \pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x), \text{ then there exists a path from } C_1 \text{ to } C_2 \text{ in } G_\Theta. \]

Proof.

We prove the contraposition.

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November 16, 2015
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Completeness

Theorem (Completeness of inheritance reasoning)

If $\pi(\Theta) \models \forall x. C_1(x) \rightarrow C_2(x)$, then there exists a path from $C_1$ to $C_2$ in $G_\Theta$.

Proof.

We prove the contraposition. Assume that there exists no such path from $C_1$ to $C_2$ in $G_\Theta$. We show that $\pi(\Theta) \not\models \forall x. C_1(x) \rightarrow C_2(x)$. 
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We prove the contraposition. Assume that there exists no such path from $C_1$ to $C_2$ in $G_\Theta$. We show that $\pi(\Theta) \not\models \forall x. C_1(x) \rightarrow C_2(x)$.

For this define an interpretation on a universe with exactly one element $d$ such that $d$ is in the interpretation of $C_1$ and in the interpretation of all concepts reachable from $C_1$ by following directed edges (and not in the interpretation of any other concept).
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This interpretation satisfies all formulae in $\pi(\Theta)$. However, it does not satisfy $\forall x. C_1(x) \rightarrow C_2(x)$. For this reason, we have $\pi(\Theta) \not\models \forall x. C_1(x) \rightarrow C_2(x)$. \qed
Semantic Networks with Instances
An extension: instances

We also want to talk about instances of concepts.

Example:

![Diagram showing instances of concepts]

- John
- Bernhard
- Studious
- Person
- Knowledgeable
- Student
- Professor
- Undergrad-Student
- Grad-Student

As formulae:
An extension: instances

We also want to talk about instances of concepts.

Example:

... as formulae:

\[\text{John inst-of Undergrad-Student} \]

\[\text{Bernhard inst-of Professor} \]
Extension of the semantics

Logical semantics:

\[ i \text{ inst-of } C \iff C(i). \]
Extension of the semantics

Logical semantics:

\[ i \text{ inst-of } C \iff C(i). \]

- **Problem 1**: Is this extension of the language conservative? That is, can we still decide \( \Theta \models C_1 \text{ isa } C_2 \) without taking formulae of the form \( i \text{ inst-of } C \) into account?
Extension of the semantics

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- **yes** (but has to be shown)
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  - yes (but has to be shown)

- **Problem 2**: Is it true: \( \Theta \models i \text{ inst-of } C \) if and only if there is a path from the node \( i \) to the node \( C \) in \( G_\Theta \)?
Extension of the semantics

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  - yes (has to be shown)

- This means, we can also use efficient graph algorithms for this extension.
Semantic Networks with Negation
A further extension: negated concepts

We now allow for negated concepts, i.e, concept terms of the form

\[ \text{not } C, \]

where \( C \) is a concept name (an atomic concept).
A further extension: negated concepts

We now allow for negated concepts, i.e., concept terms of the form

not C,

where C is a concept name (an atomic concept).

Example

Undergrad-Student isa not Grad-Student
A further extension: negated concepts

We now allow for negated concepts, i.e., concept terms of the form

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where $C$ is a concept name (an atomic concept).

Example

| Undergrad-Student isa not Grad-Student |

Logical semantics:

$$\text{not } C \iff \neg C(x)$$
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\[
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\]

where \( C \) is a concept name (an atomic concept).

**Example**

Undergrad-Student isa not Grad-Student

**Logical semantics:**

\[
\text{not } C \iff \neg C(x)
\]

**Example**

\[
C_1 \text{ isa not } C_2 \iff \forall x. C_1(x) \rightarrow \neg C_2(x).
\]
Complementing an inheritance network

Define $\overline{\alpha}$:

$$\overline{\alpha} := \begin{cases} 
\text{not } C & \text{if } \alpha = C \\
C & \text{if } \alpha = \text{not } C 
\end{cases}$$

Construct $G_\Theta$ from $\Theta$ as follows:
Complementing an inheritance network

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Construct $G_\Theta$ from $\Theta$ as follows:

- For each concept name $C$, we will have two nodes: $C$ and $\text{not } C$. 
Complementing an inheritance network

Define $\overline{\alpha}$:

$$\overline{\alpha} := \begin{cases} \text{not } C & \text{if } \alpha = C \\ C & \text{if } \alpha = \text{not } C \end{cases}$$

Construct $G_\Theta$ from $\Theta$ as follows:

- For each concept name $C$, we will have two nodes: $C$ and $\text{not } C$.
- For each formula $\alpha_1 \text{ isa } \alpha_2$, we introduce the following two edges:

$$\alpha_1 \rightarrow \alpha_2$$

$$\overline{\alpha_2} \rightarrow \overline{\alpha_1}$$
$\Theta = \{A \text{ isa not } B, \ P \text{ isa } A, \ P \text{ isa } B, \ Q \text{ isa } R, \ R \text{ isa not } A\}$
Satisfiability of an inheritance network

- Strict inheritance networks without negation are always satisfiable, i.e., they have a non-empty model (which one?)
Satisfiability of an inheritance network

- Strict inheritance networks **without negation** are always satisfiable, i.e., they have a non-empty model (which one?)
- This is no longer true when we allow for negated concepts. Consider:

\[ \text{P isa not P, not P isa P} \]

means

\[ \forall x. P(x) \rightarrow \neg P(x), \forall x. \neg P(x) \rightarrow P(x), \]

which is equivalent to

\[ \forall x. \neg P(x), \forall x. P(x). \]
Satisfiability of an inheritance network

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- \ldots i.e., this set of formulae is not satisfiable, symb. \( \Theta \models \bot \).
Satisfiability of an inheritance network

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  which is equivalent to

  \[ \forall x. \neg P(x), \ \forall x. P(x). \]

- ... i.e., this set of formulae is not satisfiable, symb. \( \emptyset \models \bot \).
- This is important to find out since in this case everything follows.
Deciding satisfiability

**Theorem (Satisfiability of strict networks with negation)**

$$\Theta \models \bot \text{ if and only if the graph } G_\Theta \text{ contains a cycle from } \alpha \text{ to } \neg \alpha \text{ and back to } \alpha.$$
Deciding satisfiability

Theorem (Satisfiability of strict networks with negation)

Θ |= ⊥ if and only if the graph G_Θ contains a cycle from α to \overline{α} and back to α.

Proof.

⇐: Adding \overline{α}_2 \rightarrow \overline{α}_1 corresponds to adding

\forall x. \neg α_2(x) \rightarrow \neg α_1(x)

when \forall x. α_1(x) \rightarrow α_2(x) is given. This is logically correct (contraposition).
Deciding satisfiability

Theorem (Satisfiability of strict networks with negation)

\[ \Theta \models \bot \text{ if and only if the graph } G_\Theta \text{ contains a cycle from } \alpha \text{ to } \neg \alpha \text{ and back to } \alpha. \]

Proof.

\( \Leftarrow \): Adding \( \neg \alpha_2 \rightarrow \neg \alpha_1 \) corresponds to adding

\[ \forall x. \neg \alpha_2 (x) \rightarrow \neg \alpha_1 (x) \]

when \( \forall x. \alpha_1 (x) \rightarrow \alpha_2 (x) \) is given. This is logically correct (contraposition). Since all directed paths in \( G_\Theta \) correspond to universally quantified implications that can be deduced from \( \pi(\Theta) \), a cycle as in the theorem implies:

\[ \forall x. \alpha (x) \rightarrow \neg \alpha (x), \forall x. \neg \alpha (x) \rightarrow \alpha (x). \]

This, however, is unsatisfiable.
Proof – continued

Proof (cont’d).

⇒: We have to show that unsatisfiability of $\Theta$ implies the existence of a cycle from some node $\alpha$ to $\overline{\alpha}$ and back to $\alpha$ in $G_\Theta$. 
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Proof – continued

Proof (cont’d).

⇒: We have to show that unsatisfiability of Θ implies the existence of a cycle from some node α to \( \overline{α} \) and back to α in \( G_Θ \).
We prove the contraposition, i.e. that the absence of any such cycle implies satisfiability.
We start with the universe \( D = \{d\} \) and then construct step-wise an interpretation for all concepts.

Proof (cont’d).

⇒: We have to show that unsatisfiability of Θ implies the existence of a cycle from some node α to α and back to α in G_Θ.
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We start with the universe $D = \{d\}$ and then construct step-wise an interpretation for all concepts.
Convention: Whenever we assign $\alpha^I = \{d\}$, then we assign $\overline{\alpha}^I = \emptyset$. 
Proof (cont’d).

\[\Rightarrow: \text{We have to show that unsatisfiability of } \Theta \text{ implies the existence of a cycle from some node } \alpha \text{ to } \overline{\alpha} \text{ and back to } \alpha \text{ in } G_\Theta.\]

We prove the contraposition, i.e. that the absence of any such cycle implies satisfiability.

We start with the universe \(\mathcal{D} = \{d\}\) and then construct step-wise an interpretation for all concepts.

Convention: Whenever we assign \(\alpha^I = \{d\}\), then we assign \(\overline{\alpha}^I = \emptyset\).

1. **Choose** an \(\alpha\) without an interpretation that has no path to \(\overline{\alpha}\).
2. **Assign** \(\alpha^I = \{d\}\) and continue to do that for all concepts \(\beta\) reachable from \(\alpha\) that do not have an interpretation.
3. **Continue** until all concepts have an interpretation.
Proof (cont’d).

⇒: We have to show that unsatisfiability of $\Theta$ implies the existence of a cycle from some node $\alpha$ to $\overline{\alpha}$ and back to $\alpha$ in $G_\Theta$.

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If there is still a concept without an interpretation, we always can find one satisfying the condition in step 1 since there is no cycle.
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3. Continue until all concepts have an interpretation.

If there is still a concept without an interpretation, we always can find one satisfying the condition in step 1 since there is no cycle.
In step 2, no concept reachable from $\alpha$ can have an empty interpretation, so the assignment does not violate any subconcept relations.
Theorem (Inheritance in strict networks with negation)

\( \Theta \models \alpha_1 \textbf{isa} \alpha_2 \) if and only if one of the following conditions is satisfied:

1. \( \Theta \models \bot \).
2. There is a path from \( \alpha_1 \) to \( \overline{\alpha_1} \) in \( G_\Theta \).
3. There is a path from \( \overline{\alpha_2} \) to \( \alpha_2 \) in \( G_\Theta \).
4. There is a path from \( \alpha_1 \) to \( \alpha_2 \) in \( G_\Theta \).

Proof (sketch).

Soundness is obvious.
isa-Reasoning

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Proof (sketch).

Soundness is obvious.
Completeness can be shown using the same argument that we used for completeness of the Satisfiability Theorem and the fact that we can start the construction process with \( \alpha_1^I = \{d\} \) and \( \overline{\alpha_2}^I = \{d\} \).

\[ \square \]
Semantic Networks with Negation and Conjunction
A final extension: conjunctions and negation

A concept description is a concept name (C), a negation of a concept name (not C) or the conjunction of concept descriptions (α₁ and α₂).

Example

(Student and not Grad-Student) isa Undergrad-Student
(Woman and Parent) isa Mother

- Logical semantics is obvious!
A final extension: conjunctions and negation

A concept description is a concept name ($C$), a negation of a concept name ($\text{not } C$) or the conjunction of concept descriptions ($\alpha_1 \text{ and } \alpha_2$).

Example

\begin{itemize}
  \item (Student \textbf{and not} Grad-Student) \textbf{isa} Undergrad-Student
  \item (Woman \textbf{and} Parent) \textbf{isa} Mother
\end{itemize}

- Logical semantics is obvious!
- Is it still possible to decide inheritance in polynomial time?
Computational complexity

**Theorem (Complexity of strict inheritance with negation and conjunction)**

The reasoning problem for strict inheritance networks with conjunction and negation is coNP-complete.

**Proof (sketch).**

We show hardness by a reduction from 3SAT.
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We show hardness by a reduction from 3SAT.

Let $D = C_1 \land \ldots \land C_n$ be formula in CNF with exactly three literals per clause (over atoms $a_i$).

Let $\sigma(C_j)$ be the following translation:

- $a_1 \lor a_2 \lor a_3 \mapsto (\neg a_1 \text{ and not } \neg a_2) \text{ isa } a_3$
- $\neg a_1 \lor a_2 \lor a_3 \mapsto (a_1 \text{ and not } a_2) \text{ isa } a_3$
- $\neg a_1 \lor \neg a_2 \lor a_3 \mapsto (a_1 \text{ and } a_2) \text{ isa } a_3$
- $\neg a_1 \lor \neg a_2 \lor \neg a_3 \mapsto (a_1 \text{ and } a_2) \text{ isa } (\neg a_3)$

Extend $\sigma$ to CNF formulae, and show that $D$ is unsatisfiable iff $\sigma(D) \models \bot$.
Conclusion

- Strict inheritance networks are easy
- Inheritance corresponds to a universally quantified implication
- If concepts are atomic, everything can be decided in poly. time
- We can deal with negation without increasing the complexity
- Conjunction and negation, however, make the reasoning problem hard
- ... as hard as propositional unsatisfiability.
Literature

. Atzeni, D. S. Parker.
Set Containment Inference and Syllogisms.