Motivation
Motivation

Basic Notions: a Reminder

Beyond NP

Oracle TMs and the Polynomial Hierarchy

Literature

Why complexity theory?

- Complexity theory can answer questions on how easy or hard a problem is
- Gives hints on what algorithms could be appropriate, e.g.:
  - algorithms for polynomial-time problems are usually easy to design
  - for NP-complete problems, backtracking and local search work well
- Gives hints on what type of algorithm will (most probably) not work
- Gives hint on what sub-problems might be interesting
Basic Notions: a Reminder
Algorithms and Turing machines

- We use Turing machines as formal models of algorithms.
- This is justified, because:
  - we assume that Turing machines can compute all computable functions.
  - the resource requirements (in terms of time and memory) of a Turing machine are only polynomially worse than other models.
- The regular type of Turing machine is the deterministic one: DTM (or simply TM).
- Often, however, we use the notion of nondeterministic TMs: NDTM.
Problems, solutions, and complexity

- A **problem** is a set of pairs \((I, A)\) of strings in \(\{0, 1\}^*\).
  - \(I\): instance; \(A\): answer
  - If all answers \(A \in \{0, 1\}\): decision problem
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- **Complexity of an algorithm**: function

  \[ T : \mathbb{N} \rightarrow \mathbb{N}, \]

  measuring the **number of basic steps** (or memory requirement) the algorithm needs to compute an answer depending on the **size** of the instance
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**Complexity of a problem**: complexity of the most efficient algorithm that solves this problem.
Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: $P$
  - Problems in $P$ are assumed to be efficiently solvable (although this might not be true if the exponent is very large)
  - In practice, this notion appears to be more often reasonable than not

- The class of problems decidable on non-deterministic Turing machines in polynomial time: $NP$

- More classes are definable using other resource bounds on time and memory
Upper and lower bounds

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  - the technical tool here is the polynomial reduction *(or any other appropriate reduction)*
  - show that some hard problem can be reduced to the problem at hand
Given languages $L_1$ and $L_2$, $L_1$ can be polynomially reduced to $L_2$, written $L_1 \leq_p L_2$, if there exists a polynomial time-computable function $f$ such that

$$x \in L_1 \iff f(x) \in L_2.$$

**Rationale:** it cannot be harder to decide $L_1$ than $L_2$
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- $L$ is hard for a class $C$ (*C*-hard) if all languages of this class can be reduced to $L$.
- $L$ is complete for $C$ (*C*-complete) if $L$ is $C$-hard and $L \in C$. 

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Problems, solutions, and complexity
Complexity classes
P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
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Literature
NP-complete problems

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- Note: \( P \) is closed under complement, in particular,

\[
P \subseteq \text{NP} \cap \text{co-NP}
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PSPACE

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PSPACE (NPSPACE) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only polynomially many tape cells.
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Some facts about PSPACE:

- PSPACE is closed under complements (as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- NP ⊆ PSPACE (because in polynomial time one can “visit” only polynomial space, i.e., NP ⊆ NPSPACE)
- It is unknown whether NP ≠ PSPACE, but it is believed that
PSPACE-completeness

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An example for a PSPACE-complete problem is the **NDFA equivalence problem**:

**Instance:** Two non-deterministic finite state automata $A_1$ and $A_2$.

**Question:** Are the languages accepted by $A_1$ and $A_2$ identical?
Other complexity classes …

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- … and for most of the classes we do not know whether the containment relationships are strict
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines

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- Usage of OTMs answers what-if questions: What if we could solve the oracle-problem efficiently?
Turing reductions

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Polynomial reducibility implies Turing reducibility, but not vice versa! NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent! Turing reducibility can also be applied to general search problems!
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Complexity classes based on Oracle TMs

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Example

Consider the **Minimum Equivalent Expression (MEE)** problem:

**Instance:** A well-formed Boolean formula $\varphi$ using the standard connectives (not $\leftrightarrow$) and a non-negative integer $k$.

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  - Oracle Turing machines
  - Turing reduction
  - Complexity classes based on OTMs
- QBF

**Literature**
The polynomial hierarchy

The complexity classes based on OTMs form an infinite hierarchy.
The polynomial hierarchy

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The polynomial hierarchy PH

\[
\begin{align*}
\Sigma_0^p &= P \\
\Sigma_{i+1}^p &= \text{NP}^{\Sigma_i^p} \\
\Pi_0^p &= P \\
\Pi_{i+1}^p &= \text{co-}\Sigma_{i+1}^p \\
\Delta_0^p &= P \\
\Delta_{i+1}^p &= P^{\Sigma_i^p}
\end{align*}
\]
The polynomial hierarchy

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### The polynomial hierarchy PH

<table>
<thead>
<tr>
<th>Class</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma^p_0$</td>
<td>$P$</td>
</tr>
<tr>
<td>$\Sigma^p_i$</td>
<td>$NP^{\Sigma^p_{i-1}}$</td>
</tr>
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- $PH = \bigcup_{i \geq 0} (\Sigma^p_i \cup \Pi^p_i \cup \Delta^p_i) \subseteq PSPACE$
- $NP = \Sigma^p_1$
- $co-NP = \Pi^p_1$
Quantified Boolean formulae: definition

- If $\varphi$ is a propositional formula, $P$ is the set of Boolean variables used in $\varphi$ and $\sigma$ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma \varphi$ is a QBF.
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- A formula $\exists x \varphi$ is true if and only if $\varphi[x/\top] \lor \varphi[x/\bot]$ is true (equivalently, $\varphi[x/\top]$ is true or $\varphi[x/\bot]$ is true).
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- This definition directly leads to an AND/OR tree traversal algorithm for evaluating QBF.
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**Example**

The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \lor y)$ are false.
The Polynomial Hierarchy: connection to QBF

Truth of QBFs with prefix $\forall \exists \ldots$ is $\Pi^p_i$-complete.
The Polynomial Hierarchy: connection to QBF

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Truth of QBFs with prefix $\exists \forall \ldots$ is $\Sigma_p^i$-complete.
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Special cases corresponding to SAT and TAUT:
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Special cases corresponding to SAT and TAUT:

- The truth of QBFs with prefix $\exists x_1^1 \ldots x_n^1$ is NP = $\Sigma^p_1$-complete.
- The truth of QBFs with prefix $\forall x_1^1 \ldots x_n^1$ is co-NP = $\Pi^p_1$-complete.
M. R. Garey and D. S. Johnson.  

C. H. Papadimitriou.  
Computational Complexity.  