The LM-cut heuristic
RPG-based relaxation heuristics seen so far,
- either admissible, but not very informative ($h_{\text{max}}$),
- or quite informative, but not admissible ($h_{\text{add}}$, $h_{\text{sa}}$, $h_{\text{FF}}$).

$\implies$ no useful relaxation heuristic for optimal planning yet.

This chapter: informative admissible relaxation heuristic ($h_{\text{LM-cut}}$).

$h_{\text{LM-cut}}$ one of the most informative admissible domain-independent heuristics currently known.
Motivation

Combination of several ideas:

- **Delete relaxation**
  - Already known from Chapter 7.
  - No repeated discussion in current chapter necessary.

- **Landmarks**
  - The central concept behind $h_{LM\text{-}cut}$.
  - Discussed first in this chapter.

- **Cost partitioning**
  - Only relevant in the non-unit-cost setting.
  - Discussed towards the end of this chapter.
Let $\Pi$ be an SAS$^+$ planning task and $s$ a state from $\Pi$.

Assume we know the following:

- In each plan starting in $s$, at least one of the operators $o_1$ and $o_2$ is applied.
- In each plan starting in $s$, at least one of the operators $o_3$ and $o_4$ is applied.
- In each plan starting in $s$, the operator $o_5$ is applied.
- In each plan starting in $s$, the operator $o_6$ is applied.
- Operators $o_1$, $o_2$, $o_3$, $o_4$, $o_5$, and $o_6$ are pairwise different.

**Question:** Does this give us a lower bound on $h^*(s)$?

**Answer:** Yes! The number of landmarks, i.e., $h^*(s) \geq 4$. 
Let $\Pi$ be an $\text{SAS}^+$ planning task and $s$ a state from $\Pi$.

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**Question:** Does this give us a lower bound on $h^*(s)$?

**Answer:** Yes! The number of landmarks, i.e., $h^*(s) \geq 4$. 
Motivation

 Technique for derivation of heuristic: landmarks.

 Question: How to compute suitable landmarks?

 For now (as long as we only consider unit-cost actions)
 suitable landmarks means disjoint landmarks.

 Counterexample for non-disjoint landmarks: Knowing that
 in each plan starting in $s$, at least one of the operators $o_1$ and $o_2$ is applied, and
 in each plan starting in $s$, at least one of the operators $o_2$ and $o_3$ is applied,
 does not imply that $h^*(s) \geq 2$, since the one-step action sequence $o_2$ might be a plan for $s$. 

Landmarks

**Definition (Landmark)**

A **landmark** of an SAS\(^+\) planning task \(\Pi\) is a set of actions \(L\) such that each plan for \(\Pi\) contains at least one action from \(L\). A landmark \(L\) for \(\Pi\) is **minimal** if no \(L' \subset L\) is a landmark for \(\Pi\).

**Note:** Landmarks in this sense are also called **disjunctive action landmarks**.

**Theorem**

*Let \(\Pi\) with initial state \(I\) be an SAS\(^+\) planning task. If there are \(n\) **disjoint** landmarks for \(\Pi\), then \(h(I) = n\) is an admissible heuristic estimate for state \(I\).*

**Proof.**

Obvious.
Landmarks

Example

\[ \langle A, I, \{ o_1, o_2, o_3, o_4, o_5 \}, \gamma \rangle \text{ with} \]

\[
A = \{ a, b, c, d, e, f, g \} \quad I = \{ a \mapsto 1 \} \cup \{ x \mapsto 0 \mid x \neq a \}
\]

\[
o_1 = \langle a, b \land c \rangle \quad o_2 = \langle a, c \land d \rangle
\]

\[
o_3 = \langle a, d \land e \rangle \quad o_4 = \langle a, e \land b \rangle
\]

\[
o_5 = \langle a, f \rangle \quad o_6 = \langle b \land c \land d \land e \land f, g \rangle
\]

\[
\gamma = g
\]

(Minimal) landmarks:

\[
\{ o_1, o_2 \} \text{ (because of } c) , \quad \{ o_2, o_3 \} \text{ (because of } d) ,
\]

\[
\{ o_3, o_4 \} \text{ (because of } e) , \quad \{ o_4, o_1 \} \text{ (because of } b) ,
\]

\[
\{ o_5 \} \text{ (because of } f) , \quad \{ o_6 \} \text{ (because of } g) .
\]
Example (ctd.)

But at most four disjoint landmarks, e.g.,
\{o_1, o_2\}, \{o_3, o_4\}, \{o_5\}, \{o_6\}.

\( h_{LM}(I) = 4 \) is admissible.
Landmarks

Theorem

Let $\Pi$ be an SAS$^+$ planning task, and let $\Pi^+$ be its delete relaxation. Let $L^+ = \{o^+ | o \in L\}$ be a landmark for $\Pi^+$. Then $L$ is also a landmark for $\Pi$.

Proof.

Let $L^+$ be a landmark for $\Pi^+$. Then every plan $\pi^+$ for $\Pi^+$ uses some action $o^+ \in L^+$. Let $\pi'$ be some plan for $\Pi$. We need to show that $\pi'$ uses some action $o \in L$. Since $\pi'$ is a plan for $\Pi$, also $\pi'^+$ is a plan for $\Pi^+$. By assumption, $\pi'^+$ must use some action $o^+ \in L^+$. But then, $\pi'$ uses action $o \in L$. \qed
Theorem

Let $\Pi$ be an SAS$^+$ planning task, and let $\Pi^+$ be its delete relaxation. Let $L^+ = \{ o^+ | o \in L \}$ be a landmark for $\Pi^+$. Then $L$ is also a landmark for $\Pi$.

$\Rightarrow$ It is sufficient to search for landmarks in the delete relaxation. This will only lead to too few discovered landmarks, not to too many.

$\Rightarrow$ Admissibility of the heuristic will be preserved.
For the rest of this chapter, we assume delete-free planning tasks $\Pi = \Pi^+$ and search for landmarks for $\Pi^+$, which gives us a good approximation of the optimal delete relaxation heuristic $h^+$. 


Landmarks
Computing Disjoint Disjunctive Action Landmarks

Naive approach:

1. Compute set $\mathcal{L} = \{L_1, \ldots, L_n\}$ of all minimal landmarks of planning task $\Pi$.
2. Compute a cardinality-maximal subset $\mathcal{L}' \subseteq \mathcal{L}$ such that all $L_i, L_j \in \mathcal{L}'$, $L_i \neq L_j$, are pairwise disjoint, and return their number, $|\mathcal{L}'|$. 

Drawbacks of naive approach: Both steps too complicated.

Simpler incomplete approach:

Compute set $\mathcal{L} = \{L_1, \ldots, L_n\}$ of some disjoint minimal landmarks for $\Pi$ incrementally.

- Compute some landmark $L_1$.
- When computing $L_{i+1}$, only consider candidates that are disjoint from all previous landmarks $L_1, \ldots, L_i$.
- Stop when no more such landmarks exist.
We implement the simpler approach by exploiting a relationship between landmarks and cuts in certain graphs:

- **Assumption:** STRIPS tasks with action costs 0 or 1.
- When computing landmark $L_{i+1}$, an action $o$ costs zero if:
  - it is a dummy action $o_s$ constructing the initial state from the unique initial proposition $s$,
  - it is a dummy action $o_t$ constructing the unique dummy goal proposition $t$ from the actual goal propositions, or
  - it has already been included in one of the previous landmarks $L_1, \ldots, L_i$, i.e., it has already been accounted for in the heuristic computation.
To that end, in the algorithm we will present, action cost values will be iteratively decremented.

- In the first iteration, we have action costs $c_1(o_s) = c_1(o_t) = 0$, and $c_1(o) = 1$ for all other actions $o$.
- Cost functions in later iterations $i + 1$ are denoted by $c_{i+1}$ and will differ from $c_i$ in that costs of actions used in $L_i$ are set to zero.
Assumptions and Definitions

Definition (Precondition-choice function)
A precondition-choice function (pcf) is a function $D$ that maps each action into one of its preconditions. (We assume that each action has at least one precondition.)

Definition (Justification graph)
The justification graph for a pcf $D$, denoted by $G(D)$, is a directed graph whose vertices are the propositions and which has an edge $(p, q)$ labeled with $o$ iff the action $o$ adds $q$ and $D(o) = p$. 
Assumptions and Definitions

Definition (Cut)

For two nodes \( s \) and \( t \) in a justification graph, an \( s-t \) cut in that justification graph is a subset \( C \) of its edges such that all paths from \( s \) to \( t \) use an edge from \( C \).

When \( s \) and \( t \) are clear, we simply call \( C \) a cut.

Theorem (Cuts correspond to landmarks)

Let \( C \) be a cut in a justification graph for an arbitrary pcf. Then the edge labels for \( C \) are a landmark.
### Assumptions and Definitions

**Definition (\(h_{\text{max}}\) costs of atoms)**

Given a fixed initial state \(s\) and an action cost function \(c\), the \(h_{\text{max}}\) cost of an atom \(a\), denoted by \(h_{\text{max}}^c(a)\), is the value the RPG proposition node for atom \(a\) in the last RPG layer is labeled with after the RPG computation (with layer 0 initialized with state \(s\) and action costs given by \(c\)) has converged/stabilized.

Intuitively, \(h_{\text{max}}^c(a)\) is the cost of making \(a\) true under parallel relaxed semantics, maximizing over precondition costs. For unit-costs tasks, \(h_{\text{max}}^c(a)\) would be the index of the first RPG layer in the RPG seeded with \(s\) where \(a\) becomes true.
LM-cut Heuristic: Motivation

- In general **exponentially many pcfs**, i.e., we cannot compute all relevant landmarks.
- The **LM-cut heuristic** is a method to compute pcfs and cuts in a goal-directed way.
- Efficient partitioning of actions into cuts.
  - Currently best admissible planning heuristic
Pseudocode of LM-cut heuristic

Initialize $h = 0$ and $i = 1$.

Step 1. Compute $h_{\text{max}}^c(a)$ values for every atom $a \in A$. Terminate if $h_{\text{max}}^c(t) = 0$.

Step 2. Compute pcf $D_i$: Modify actions by keeping only one proposition in the precondition of each action: a proposition maximizing $h_{\text{max}}^c$, breaking ties arbitrarily.

Step 3. Construct justification graph $G_i$ of $D_i$: Vertices are the propositions; for each action $o = \langle p, q_1 \land \ldots \land q_k \rangle$ and each $j = 1, \ldots, k$, there is an edge from $p$ to $q_j$ with cost $c_i(o)$ and label $o$.

Step 4. …
Pseudocode of LM-cut heuristic (ctd.)

Step 4. Construct an $s$-$t$-cut $C_i = (V_i^0, V_i^* \cup V_i^b)$ of $G_i$ as follows: $V_i^*$ contains all propositions from which $t$ can be reached through a zero-cost path, $V_i^0$ contains all propositions reachable from $s$ without passing through some propositions in $V_i^*$, and $V_i^b$ contains all remaining propositions. Clearly, $s \in V_i^0$ and $t \in V_i^*$.

Step 5. Determine disjunctive action landmark: Let $L_i$ be the set of labels of the edges that cross the cut $C_i$ (i.e., lead from $V_i^0$ to $V_i^*$).

Step 6. Decrease action costs: Define $c_{i+1}(o) := c_i(o)$ if $o \notin L_i$, and $c_{i+1}(o) := 0$ if $o \in L_i$.


Step 8. Set $i := i + 1$ and go to Step 1.
Example

Adaptation/simplification of running example from Chapter 8: planning task \( \langle A, I, \{o_s, o_1, o_2, o_3, o_4, o_t\}, \gamma \rangle \) with

\[
A = \{s, a, b, c, d, e, f, g, h, t\}
\]

\[
I = \{s \mapsto 1, a \mapsto 0, b \mapsto 0, c \mapsto 0, d \mapsto 0, e \mapsto 0, f \mapsto 0, g \mapsto 0, h \mapsto 0, t \mapsto 0\}
\]

\[
o_s = \langle s, a \land c \land d \rangle
\]

\[
o_1 = \langle c \land d, b \rangle
\]

\[
o_2 = \langle a \land b, e \rangle
\]

\[
o_3 = \langle a, f \rangle
\]

\[
o_4 = \langle f, g \land h \rangle
\]

\[
o_t = \langle e \land g \land h, t \rangle
\]

\[
\gamma = t
\]
Example

- **Cheapest sequential (relaxed) plan:** $\langle o_s, o_1, o_2, o_3, o_4, o_t \rangle$ with cost $h^+(l) = 4$ (recall that $o_s$ and $o_t$ cost nothing).

- **Parallel (relaxed) plan witnessing $h_{\text{max}}(l) = 2$:**
  
  $\langle \{ o_s \}, \{ o_1, o_3 \}, \{ o_2, o_4 \}, \{ o_t \} \rangle$.

**Our aim:** Get closer to $h^+(l) = 4$ using $h_{\text{LM-cut}}$ than using $h_{\text{max}}$. 
Example: Iteration 1

<table>
<thead>
<tr>
<th>prop p</th>
<th>s</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{\text{max}}^c(p)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>action o</th>
<th>$o_s$</th>
<th>$o_1$</th>
<th>$o_2$</th>
<th>$o_3$</th>
<th>$o_4$</th>
<th>$o_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pcf $D_1(o)$</td>
<td>s</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>f</td>
<td>g</td>
</tr>
</tbody>
</table>

$o_s[0] = \langle s, a \land c \land d \rangle$

$o_1[1] = \langle c \land d, b \rangle$

$o_2[1] = \langle a \land b, e \rangle$

$o_3[1] = \langle a, f \rangle$

$o_4[1] = \langle f, g \land h \rangle$

$o_t[0] = \langle e \land g \land h, t \rangle$

$L_1 = \{ o_4 \}$, $h_{\text{LM-cut}}(l)$ so far = 1
Example: Iteration 2

<table>
<thead>
<tr>
<th>prop $p$</th>
<th>$s$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{max}^c(p)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>action $o$</th>
<th>$o_s$</th>
<th>$o_1$</th>
<th>$o_2$</th>
<th>$o_3$</th>
<th>$o_4$</th>
<th>$o_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>pcf $D_2(o)$</td>
<td>$s$</td>
<td>$c$</td>
<td>$b$</td>
<td>$a$</td>
<td>$f$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

\[ L_2 = \{ o_2 \}, \quad h_{LM-cut}(I) \text{ so far} = 2 \]
Example: Iteration 3

<table>
<thead>
<tr>
<th>prop $p$</th>
<th>$s$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
<th>$h$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{\text{max}}^c(p)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>action $o$</th>
<th>$o_s$</th>
<th>$o_1$</th>
<th>$o_2$</th>
<th>$o_3$</th>
<th>$o_4$</th>
<th>$o_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{pcf } D_3(o)$</td>
<td>$s$</td>
<td>$c$</td>
<td>$b$</td>
<td>$a$</td>
<td>$f$</td>
<td>$g$</td>
</tr>
</tbody>
</table>

$o_s[0] = \langle s, a \land c \land d \rangle$

$o_1[1] = \langle c \land d, b \rangle$

$o_2[0] = \langle a \land b, e \rangle$

$o_3[1] = \langle a, f \rangle$

$o_4[0] = \langle f, g \land h \rangle$

$o_t[0] = \langle e \land g \land h, t \rangle$

$L_3 = \{o_3\}$, $h_{\text{LM-cut}}(I)$ so far = 3
Example: Iteration 4

<table>
<thead>
<tr>
<th>prop p</th>
<th>s</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{\text{max}}^c(p)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>action o</th>
<th>o_s</th>
<th>o_1</th>
<th>o_2</th>
<th>o_3</th>
<th>o_4</th>
<th>o_t</th>
</tr>
</thead>
<tbody>
<tr>
<td>pcf $D_4(o)$</td>
<td>s</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>f</td>
<td>e</td>
</tr>
</tbody>
</table>

$o_s[0] = \langle s, a \land c \land d \rangle$
o_1[1] = \langle c \land d, b \rangle$
o_2[0] = \langle a \land b, e \rangle$
o_3[0] = \langle a, f \rangle$
o_4[0] = \langle f, g \land h \rangle$
o_t[0] = \langle e \land g \land h, t \rangle$

$L_4 = \{ o_1 \}$, $h_{\text{LM-cut}}(l)$ so far = 4
Example: Iteration 5

<table>
<thead>
<tr>
<th>propp</th>
<th>s</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{\text{max}}^c(p)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| action o | $o_s$ | $o_1$ | $o_2$ | $o_3$ | $o_4$ | $o_t$ |
| action o | s | c | b | a | f | g |

- $o_s[0] = \langle s, a \land c \land d \rangle$
- $o_1[0] = \langle c \land d, b \rangle$
- $o_2[0] = \langle a \land b, e \rangle$
- $o_3[0] = \langle a, f \rangle$
- $o_4[0] = \langle f, g \land h \rangle$
- $o_t[0] = \langle e \land g \land h, t \rangle$

$h_{\text{LM-cut}}(l) = 4$
Theorem

The LM-cut heuristic never overestimates $h^+$, i.e., it is admissible.

Proof sketch

- From every landmark found, at least one operator has to be applied in any relaxed plan.
- Each found landmark is counted only once and there is no overlap in operators used in landmarks, i.e., the landmarks that are found are disjoint (operator costs for all operators in a “used” landmark are reset to zero).
- Therefore, we count at most as many landmarks as there are operators in a shortest relaxed plan.
Remark: $h_{LM\text{-cut}}$ can be generalized to planning tasks with non-unit costs.

Instead of setting operator costs to zero, decrease costs of all operators in landmark by the minimal cost of any operator in the landmark. This effectively leads to a cost partitioning of operator costs between landmarks: An operator can be (partly) counted in more than one landmark, but the sum of the weights it is counted with will not exceed its true cost.

Instead of incrementing heuristic value by one in each step, increase it by minimal cost of any operator in the landmark.

Then, $h_{LM\text{-cut}}$ is still admissible. Proof via cost-partitioning argument.
Outlook: Non-unit-cost tasks

Example

Iter. 1: $D(t) = a \leadsto L_1 = \{o_2, o_3\}$ [4]

$o_1[3] = \langle s, a \land b \rangle$

$o_2[4] = \langle s, a \land c \rangle$

$o_3[5] = \langle s, b \land c \rangle$

$o_4[0] = \langle a \land b \land c, t \rangle$
Outlook: Non-unit-cost tasks

Example

Iter. 1: \( D(t) = a \leadsto L_1 = \{ o_2, o_3 \} [4] \leadsto h_{LM-cut}(l) := 4 \)

\[
\begin{align*}
  o_1[3] & = \langle s, a \land b \rangle \\
  o_2[0] & = \langle s, a \land c \rangle \\
  o_3[1] & = \langle s, b \land c \rangle \\
  o_4[0] & = \langle a \land b \land c, t \rangle
\end{align*}
\]
Outlook: Non-unit-cost tasks

Example

Iter. 2: \( D(t) = b \sim L_2 = \{o_1, o_3\} \) [1]

\[
\begin{align*}
o_1[3] &= \langle s, a \land b \rangle \\
o_2[0] &= \langle s, a \land c \rangle \\
o_3[1] &= \langle s, b \land c \rangle \\
o_4[0] &= \langle a \land b \land c, t \rangle
\end{align*}
\]
Example

Iter. 2: $D(t) = b \sim L_2 = \{o_1, o_3\}$ [1] $\sim h_{\text{LM-cut}}(l) := 4 + 1 = 5$

\[ o_1[2] = \langle s, a \land b \rangle \]
\[ o_2[0] = \langle s, a \land c \rangle \]
\[ o_3[0] = \langle s, b \land c \rangle \]
\[ o_4[0] = \langle a \land b \land c, t \rangle \]
Outlook: Non-unit-cost tasks

Example

Iter. 3: $h_{\text{max}}(t) = 0 \Rightarrow \text{done!} \Rightarrow h_{\text{LM-cut}}(I) = 5$

$$o_1[2] = \langle s, a \land b \rangle$$
$$o_2[0] = \langle s, a \land c \rangle$$
$$o_3[0] = \langle s, b \land c \rangle$$
$$o_4[0] = \langle a \land b \land c, t \rangle$$
Outlook: Non-unit-cost tasks

Remark: The costs of $o_3$ (i.e., 5) were partitioned as follows:
- 4 cost units were used in the cost of $L_1$, and
- 1 cost unit was used in the cost of $L_2$.

Without this cost partitioning, we would have only found $L_1$ and counted it at a cost of 4. Landmark $L_2$ would not have been considered, since it is not disjoint from $L_1$.

Thus, we would have arrived at an unnecessarily low value $h_{\text{LM-cut}}(I) = 4$ instead of $h_{\text{LM-cut}}(I) = 5$. 
Summary

- **Landmarks** are sets of actions such that each plan contains at least one of these actions.
- **Cuts in justification graphs** are a very general method to find landmarks.
- The **LM-cut heuristic** is an efficient admissible heuristic based on landmarks and cuts.
- It combines **delete relaxation, landmarks, and cost partitioning**.