1 Motivation

Reasoning problems:
- **Satisfiability** or **subsumption** of concept descriptions
- **Satisfiability** or **instance relation** in ABoxes

Solving techniques presented in this chapter:
- **Structural subsumption algorithms**
  - Normalization of concept descriptions and structural comparison
  - Very fast, but can only be used for small DLs
- **Tableau algorithms**
  - Similar to modal tableau methods
  - Often the method of choice
2 Structural Subsumption Algorithms

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- ABox Reasoning

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Structural subsumption algorithms

In what follows we consider the rather small logic $\mathcal{FL}^e$:

$C \sqcap D$

$\forall r.C$

$\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

$\forall r.C \sqcap \forall r.D \rightarrow \forall r.(C \sqcap D)$.

2. Compare all conjuncts with each other.
   For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.

Example

Example

$D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.Human} \sqcap \forall \text{has-child.} \exists \text{has-child}$

$C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.(Human } \sqcap \text{Female} \sqcap \exists \text{has-child)}$

Check: $C \sqsubseteq D$

1. Collect value restrictions in $D$:
   $\ldots \exists \text{has-child.(Human } \sqcap \exists \text{has-child)}$

2. Compare:
   1. For Human in $D$, we have Human in $C$.
   2. For $\exists \text{has-child}$ in $D$, we have $\exists \text{has-child}$ in $C$.
   3. For $\forall \text{has-child(...)}$ in $D$, we have Human and $\exists \text{has-child}$ in $C$.

$\Rightarrow C \sqsubseteq D$

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Subsumption algorithm

$\text{SUB}(C,D)$ algorithm:

1. Reorder terms (using commutativity, associativity and value restriction law):

$C = \prod A_i \sqcap \prod \exists r_j \sqcap \forall r_k : C_k$

$D = \prod B_i \sqcap \prod \exists s_m \sqcap \forall s_n : D_n$

2. For each $B_i$ in $D$, is there an $A_i$ in $C$ with $A_i = B_i$?

3. For each $\exists s_m$ in $D$, is there an $\exists r_j$ in $C$ with $s_m = r_j$?

4. For each $\forall s_n : D_n$ in $D$, is there a $\forall r_k : C_k$ in $C$ such that $s_n = r_k$ and $C_k \sqsubseteq D_n$ (i.e., check $\text{SUB}(C_k,D_n)$)?

$\Rightarrow C \sqsubseteq D$ iff all questions are answered positively.
Soundness

**Theorem (Soundness)**

\[ \text{SUB}(C, D) \Rightarrow C \sqsubseteq D \]

**Proof sketch.**

Reordering of terms step (1):

1. Commutativity and associativity are trivial
2. Value restriction law. We show: \((\forall r. (C \cap D)) = (\forall r. C \cap \forall r. D)\)
   
   Assume \(d \in (\forall r. (C \cap D))\).
   
   If there is no \(e \in D\) with \((d, e) \in r\), it follows trivially that \(d \in (\forall r. C \cap \forall r. D)\).
   
   If there is an \(e \in D\) with \((d, e) \in r\), it follows \(e \in (C \cap D)\).
   
   Since \(e\) is arbitrary, we have \(d \in (\forall r. C)\) and \(d \in (\forall r. D)\).
   
   i.e., \((\forall r. (C \cap D)) \subseteq (\forall r. C \cap \forall r. D)\).

   The other direction is similar.

Steps (2+3+4): Induction on the nesting depth of \(\forall\)-expressions.

Completeness

**Theorem (Completeness)**

\[ C \sqsubseteq D \Rightarrow \text{SUB}(C, D) \]

**Proof idea.**

One shows the contrapositive:

\(\neg \text{SUB}(C, D) \Rightarrow C \not\sqsubseteq D\)

**Idea:** If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element \(d\) such that \(d \in C\), but \(d \not\in D\).

Generalizing the algorithm

Extensions of \(\mathcal{FL}^-\) by

- \(\neg A\) (atomic negation),
- \((\leq nr), (\geq nr)\) (cardinality restrictions),
- \(r \circ s\) (role composition)

do not lead to any problems.

However: If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.

More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

Reason: Subsumption for \(\mathcal{FL}^- + \exists r. C\) is \(\text{NP}\)-hard (Nutt).

ABox reasoning

**Idea:** Abstraction + classification

- **Complete** ABox by propagating value restrictions to role fillers.
- Compute for each object its most specialized concepts.
- These can then be handled using the ordinary subsumption algorithm.
3 Tableau Subsumption Method

- Example
- Reductions: Unfolding & Unsatisfiability
- Model Construction
- Equivalences & NNF
- Constraint Systems
- Transforming Constraint Systems
- Invariances
- Soundness and Completeness
- Space Complexity
- ABox Reasoning

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Tableau method

Logic $\mathcal{ALC}$:
- $C \sqcap D$
- $C \sqcup D$
- $\neg C$
- $\forall x. C$
- $\exists x. C$

Idea: Decide (un-)satisfiability of a concept description $C$ by trying to systematically construct a model for $C$. If that is successful, $C$ is satisfiable. Otherwise, $C$ is unsatisfiable.

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Example: Subsumption in a TBox

Example
TBox:

Hermaphrodite $\equiv$ Male $\sqcap$ Female
Parent-of-sons-and-daughters $\equiv$ $\exists$ has-child.Male $\sqcap$ $\exists$ has-child.Female
Parent-of-hermaphrodite $\equiv$ $\exists$ has-child.Hermaphrodite

Query:
Parent-of-sons-and-daughters $\subseteq T$
Parent-of-hermaphrodites

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Reductions

1. Unfolding:
   $\exists$ has-child.Male $\sqcap$ $\exists$ has-child.Female $\sqsubseteq \exists$ has-child.(Male $\sqcap$ Female)
2. Reduction to unsatisfiability: Is the concept $\exists$ has-child.Male $\sqcap$ $\exists$ has-child.Female $\sqcap$ $\neg$ $\exists$ has-child.(Male $\sqcap$ Female) unsatisfiable?
3. Negation normal form (move negations inside):
   $\exists$ has-child.Male $\sqcap$ $\exists$ has-child.Female $\sqcap$ $\forall$ has-child.(~Male $\sqcup$ ~Female)
4. Try to construct a model

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Assumption: There exists an object $x$ in the interpretation of our concept:

$$x \in (\exists \ldots)^I$$

This implies that $x$ is in the interpretation of all conjuncts:

$$x \in (\exists \text{has-child}\text{.Male})^I$$
$$x \in (\exists \text{has-child}\text{.Female})^I$$
$$x \in (\forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}))^I$$

This implies that there should be objects $y$ and $z$ such that $(x,y) \in \text{has-child}^I, (x,z) \in \text{has-child}^I, y \in \text{Male}^I$ and $z \in \text{Female}^I$, and ...
Model construction (5)

```plaintext
x : ∃has-child. Male
x : ∃has-child. Female
x : ∀has-child. (¬Male ⊔ ¬Female)
y : ¬Female
z : ¬Male
```

⇝ Model constructed!

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Tableau method (1): NNF

We write: \( C \equiv D \) iff \( C \sqsubseteq D \) and \( D \sqsubseteq C \). Now we have the following equivalences:

\[
\neg(C \sqcap D) \equiv \neg C \sqcup \neg D \\
\neg(\forall r.C) \equiv \exists r.\neg C \\
\neg(\exists r.C) \equiv \forall r.\neg C \\
\neg\neg C \equiv C
\]

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).

**Theorem (NNF)**

The negation normal form of an \( \mathcal{ALC} \) concept can be computed in polynomial time.

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Tableau method (2): Constraint systems

A constraint is a syntactical object of the form:

\[
x : C \text{ or } xry,
\]

where \( C \) is a concept description in NNF, \( r \) is a role name, and \( x \) and \( y \) are variable names.

Let \( I \) be an interpretation with universe \( D \). An \( I \)-assignment \( \alpha \) is a function that maps each variable symbol to an object of the universe \( D \).

A constraint \( x : C (xry) \) is satisfied by an \( I \)-assignment \( \alpha \) if \( \alpha(x) \in C \) (resp. \( \alpha(x), \alpha(y) \in r \)).

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Tableau method (3): Constraint systems

**Definition**

A constraint system \( S \) is a finite, non-empty set of constraints. An \( I \)-assignment \( \alpha \) satisfies \( S \) if \( \alpha \) satisfies each constraint in \( S \). \( S \) is satisfiable if there exist \( I \) and \( \alpha \) such that \( \alpha \) satisfies \( S \).

**Theorem**

An \( \mathcal{ALC} \) concept \( C \) in NNF is satisfiable if and only if the system \( \{ x : C \} \) is satisfiable.

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Tableau method (4): Transforming constraint systems

**Transformation rules:**

1. \( S \rightarrow \{ x: C_1, x: C_2 \} \cup S \)
   - if \( (x: C_1 \cap C_2) \in S \) and either \((x: C_1) \in S \) or \((x: C_2) \in S \) or both are not in \( S \).
2. \( S \rightarrow \{ x: D \} \cup S \)
   - if \((x: C_1 \cup C_2) \in S \) and neither \((x: C_1) \in S \) nor \((x: C_2) \in S \), and \( D = C_1 \) or \( D = C_2 \).
3. \( S \rightarrow \{ x r y : y : C \} \cup S \)
   - if \((x: \exists r.C) \in S \), \( y \) is a fresh variable, and there is no \( z \) s.t. \((x r z) \in S \) and \((z: C) \in S \).
4. \( S \rightarrow \{ y : C \} \cup S \)
   - if \((x: \forall r.C), (x r y) \in S \) and \((y: C) \notin S \).

**Notice:** Deterministic rules (1,3,4) vs. non-deterministic (2).

Generating rules (3) vs. non-generating (1,2,4).

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Tableau method (5): Invariances

**Theorem (Invariance)**

Let \( S \) and \( T \) be constraint systems.

1. If \( T \) has been generated by applying a deterministic rule to \( S \), then \( S \) is satisfiable if and only if \( T \) is satisfiable.
2. If \( T \) has been generated by applying a non-deterministic rule to \( S \), then \( S \) is satisfiable if \( T \) is satisfiable.

Furthermore, if a non-deterministic rule can be applied to \( S \), then it can be applied such that \( S \) is satisfiable if and only if the resulting system \( T \) is satisfiable.

**Theorem (Termination)**

Let \( C \) be an \( {\text{ALC}} \) concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system \( \{ x : C \} \).

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Space requirements

Because the tableau method is non-deterministic (\( \rightarrow \) rule), there could be exponentially many closed constraint systems in the end.

Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

**Example**

\[
\begin{align*}
\exists r.A & \quad \exists r.B \\
\forall r. (\exists r.A & \quad \exists r.B) \\
\forall r. (\quad \exists r.A & \quad \exists r.B)
\end{align*}
\]

However: One can modify the algorithm so that it needs only polynomial space.

**Idea:** One can modify the algorithm so that it needs only polynomial space.

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A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form \( x: A \) and \( x: \neg A \), where \( A \) is a concept name.

**Theorem (Soundness and Completeness)**

A closed constraint system is satisfiable if and only if it does not contain a clash.

**Proof idea.**

\( \Rightarrow \): obvious. \( \Leftarrow \): Construct a model by using the concept labels.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):

- Normalize and unfold and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in $\mathcal{ALC}$ we do not need this because we are never forced to identify two objects.

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Literature I

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- Hector J. Levesque and Ronald J. Brachman.
  Expressiveness and tractability in knowledge representation and reasoning.
- Manfred Schmidt-Schauß and Gert Smolka.
  Attributive concept descriptions with complements.
- Bernhard Hollunder and Werner Nutt.
  Subsumption Algorithms for Concept Languages.

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Literature II

- F. Baader and U. Sattler.
  An Overview of Tableau Algorithms for Description Logics.
  Practical Reasoning for Very Expressive Description Logics.

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