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1 Motivation
Reasoning problems & algorithms

Reasoning problems:
- **Satisfiability or subsumption** of concept descriptions
- **Satisfiability** or instance relation in ABoxes

Solving techniques presented in this chapter:
- **Structural subsumption algorithms**
  - Normalization of concept descriptions and structural comparison
  - very fast, but can only be used for small DLs
- **Tableau algorithms**
  - Similar to modal tableau methods
  - Often the method of choice
2 Structural Subsumption Algorithms

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Structural subsumption algorithms

In what follows we consider the rather small logic $\mathcal{FL}^-$:

- $C \sqcap D$
- $\forall r. C$
- $\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

   $$\forall r. C \sqcap \forall r. D \rightarrow \forall r. (C \sqcap D).$$

2. Compare all conjuncts with each other.
   For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.
Example

Example

\[ D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}.\text{Human} \sqcap \]
\[ \forall \text{has-child}.\exists \text{has-child} \]
\[ C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \]
\[ \forall \text{has-child}.(\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child}) \]

Check: \( C \subseteq D \)

1. Collect value restrictions in \( D \):
   \[ \ldots \forall \text{has-child}.(\text{Human} \sqcap \exists \text{has-child}) \]

2. Compare:
   1. For \( \text{Human} \) in \( D \), we have \( \text{Human} \) in \( C \).
   2. For \( \exists \text{has-child} \) in \( D \), we have \( \exists \text{has-child} \) in \( C \).
   3. For \( \forall \text{has-child}.(...) \) in \( D \), we have
      \( \text{Human} \) and \( \exists \text{has-child} \) in \( C \).

\( C \) is subsumed by \( D \)!
Subsumption algorithm

SUB(C, D) algorithm:

1. Reorder terms (using commutativity, associativity and value restriction law):

   \[
   C = \bigcap A_i \cap \bigcap \exists r_j \cap \bigcap \forall r_k : C_k \\
   D = \bigcap B_l \cap \bigcap \exists s_m \cap \bigcap \forall s_n : D_n
   \]

2. For each \( B_l \) in \( D \), is there an \( A_i \) in \( C \) with \( A_i = B_l \)?
3. For each \( \exists s_m \) in \( D \), is there an \( \exists r_j \) in \( C \) with \( s_m = r_j \)?
4. For each \( \forall s_n : D_n \) in \( D \), is there a \( \forall r_k : C_k \) in \( C \) such that \( s_n = r_k \) and \( C_k \sqsubseteq D_n \) (i.e., check \( \text{SUB}(C_k, D_n) \))?

\[\Rightarrow\] \( C \sqsubseteq D \) iff all questions are answered positively.
Soundness

Theorem (Soundness)

\[ SUB(C, D) \implies C \sqsubseteq D \]

Proof sketch.

Reordering of terms step (1):

1. Commutativity and associativity are trivial

2. Value restriction law. We show: \((\forall r.(C \sqcap D))^I = (\forall r.C \sqcap \forall r.D)^I\)
   
   Assume \(d \in (\forall r.(C \sqcap D))^I\).
   
   If there is no \(e \in D\) with \((d,e) \in r^I\) it follows trivially that \(d \in (\forall r.C \sqcap \forall r.D)^I\).
   
   If there is an \(e \in D\) with \((d,e) \in r^I\) it follows \(e \in (C \sqcap \forall r.D)^I = C^I \sqcap D^I\).
   
   Since \(e\) is arbitrary, we have \(d \in (\forall r.C)^I\) and \(d \in (\forall r.D)^I\), i.e., \((\forall r.(C \sqcap D))^I \subseteq (\forall r.C \sqcap \forall r.D)^I\).
   
   The other direction is similar.

Steps (2+3+4): Induction on the nesting depth of \(\forall\)-expressions.
Completeness

Theorem (Completeness)

\[ C \sqsubseteq D \Rightarrow \text{SUB}(C, D). \]

Proof idea.

One shows the contrapositive:

\[ \neg \text{SUB}(C, D) \Rightarrow C \not\sqsubseteq D \]

Idea: If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element \( d \) such that

\[ d \in C^\mathcal{I}, \text{ but } d \notin D^\mathcal{I}. \]
Generalizing the algorithm

Extensions of $\mathcal{FL}^-$ by

- $\neg A$ (atomic negation),
- $(\leq nr)$, $(\geq nr)$ (cardinality restrictions),
- $r \circ s$ (role composition)

do not lead to any problems.

**However**: If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.

**More precisely**: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

**Reason**: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).
**Idea:** Abstraction + classification

- **Complete** ABox by propagating value restrictions to role fillers.
- Compute for each object its **most specialized concepts**.
- These can then be handled using the ordinary subsumption algorithm.
3 Tableau Subsumption Method

- Example
- Reductions: Unfolding & Unsatisfiability
- Model Construction
- Equivalences & NNF
- Constraint Systems
- Transforming Constraint Systems
- Invariances
- Soundness and Completeness
- Space Complexity
- ABox Reasoning
Tableau method

Logic \( \mathcal{ALC} \):  
- \( C \sqcap D \)  
- \( C \sqcup D \)  
- \( \neg C \)  
- \( \forall r.C \)  
- \( \exists r.C \)  

Idea: Decide (un-)satisfiability of a concept description \( C \) by trying to systematically construct a model for \( C \). If that is successful, \( C \) is satisfiable. Otherwise, \( C \) is unsatisfiable.
Example: Subsumption in a TBox

Example

TBox:

\[
\begin{align*}
\text{Hermaphrodite} & \equiv \text{Male} \sqcap \text{Female} \\
\text{Parent-of-sons-and-daughters} & \equiv \\
& \exists \text{has-child.Male} \sqcap \exists \text{has-child.Female} \\
\text{Parent-of-hermaphrodites} & \equiv \exists \text{has-child.Hermaphrodite}
\end{align*}
\]

Query:

\[
\begin{align*}
\text{Parent-of-sons-and-daughters} & \sqsubseteq \mathcal{T} \\
\text{Parent-of-hermaphrodites}
\end{align*}
\]
Reductions

1. **Unfolding:**
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \]
   \[ \sqsubseteq \exists \text{has-child}.(\text{Male} \sqcap \text{Female}) \]

2. **Reduction to unsatisfiability:** Is the concept
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \]
   \[ \neg \exists \text{has-child}.(\text{Male} \sqcap \text{Female}) \]
   unsatisfiable?

3. **Negation normal form** (move negations inside):
   \[ \exists \text{has-child}. \text{Male} \sqcap \exists \text{has-child}. \text{Female} \sqcap \]
   \[ \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]

4. **Try to construct a model**
Model construction (1)

1 Assumption: There exists an object \( x \) in the interpretation of our concept:

\[
x \in (\exists \ldots)^I
\]

2 This implies that \( x \) is in the interpretation of all conjuncts:

\[
x \in (\exists \text{has-child}.\text{Male})^I
\]
\[
x \in (\exists \text{has-child}.\text{Female})^I
\]
\[
x \in (\forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}))^I
\]

3 This implies that there should be objects \( y \) and \( z \) such that

\[
(x, y) \in \text{has-child}^I, (x, z) \in \text{has-child}^I, y \in \text{Male}^I \text{ and } z \in \text{Female}^I, \text{ and } ...
\]
Model construction (2)

\[ x : \exists \text{has-child}. \text{Male} \]
\[ x : \exists \text{has-child}. \text{Female} \]
Model construction (3)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]
\[ x : \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female}) \]
Model construction (4)

\[
\begin{align*}
  x & : \exists \text{has-child. Male} \\
  x & : \exists \text{has-child. Female} \\
  x & : \forall \text{has-child.} (\neg \text{Male} \sqcup \neg \text{Female}) \\
  y & : \neg \text{Male}
\end{align*}
\]

\[
\begin{array}{c}
\text{has-child} \\
\text{has-child}
\end{array}
\]

\[
\begin{array}{c}
y \\
\text{Male}
\end{array} \quad \quad \quad
\begin{array}{c}
z \\
\text{Female}
\end{array}
\]

\[
\begin{array}{c}
\neg \text{Male or } \neg \text{Female} \\
\neg \text{Male or } \neg \text{Female}
\end{array}
\]

\[
\neg \text{Male \quad \text{Contradiction}}
\]

February 10, 2014 Nebel, Wölfli, Hué – KRR
Model construction (5)

\[
\begin{align*}
x & : \exists \text{has-child}.\text{Male} \\
{x} & : \exists \text{has-child}.\text{Female} \\
{x} & : \forall \text{has-child}. (\neg \text{Male} \sqcup \neg \text{Female}) \\
y & : \neg \text{Female} \\
z & : \neg \text{Male}
\end{align*}
\]

\[\Rightarrow \]

Model constructed!

\[\Rightarrow \text{Model constructed!} \]
Tableau method (1): NNF

We write: $C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$. Now we have the following equivalences:

\[
\neg(C \land D) \equiv \neg C \lor \neg D \\
\neg(C \lor D) \equiv \neg C \land \neg D \\
\neg(\forall r. C) \equiv \exists r. \neg C \\
\neg(\exists r. C) \equiv \forall r. \neg C \\
\neg\neg C \equiv C
\]

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).

**Theorem (NNF)**

*The negation normal form of an $\mathcal{ALC}$ concept can be computed in polynomial time.*
A constraint is a syntactical object of the form:

$$x : C \quad \text{or} \quad x r y,$$

where $C$ is a concept description in NNF, $r$ is a role name, and $x$ and $y$ are variable names.

Let $\mathcal{I}$ be an interpretation with universe $\mathcal{D}$. An $\mathcal{I}$-assignment $\alpha$ is a function that maps each variable symbol to an object of the universe $\mathcal{D}$.

A constraint $x : C \ (x r y)$ is satisfied by an $\mathcal{I}$-assignment $\alpha$ if $\alpha(x) \in C^{\mathcal{I}}$ (resp. $(\alpha(x), \alpha(y)) \in r^{\mathcal{I}}$).
Tableau method (3): Constraint systems

Definition

A constraint system $S$ is a finite, non-empty set of constraints. An $\mathcal{I}$-assignment $\alpha$ satisfies $S$ if $\alpha$ satisfies each constraint in $S$. $S$ is satisfiable if there exist $\mathcal{I}$ and $\alpha$ such that $\alpha$ satisfies $S$.

Theorem

An $\mathcal{ALC}$ concept $C$ in NNF is satisfiable if and only if the system $\{x : C\}$ is satisfiable.
Tableau method (4): Transforming constraint systems

Transformation rules:

1. \( S \rightarrow \cap \{ x : C_1, x : C_2 \} \cup S \)
   if \( (x : C_1 \cap C_2) \in S \) and either \( (x : C_1) \) or \( (x : C_2) \) or both are not in \( S \).

2. \( S \rightarrow \cup \{ x : D \} \cup S \)
   if \( (x : C_1 \cup C_2) \in S \) and neither \( (x : C_1) \in S \) nor \( (x : C_2) \in S \) and \( D = C_1 \) or \( D = C_2 \).

3. \( S \rightarrow \exists \{ x r y, y : C \} \cup S \)
   if \( (x : \exists r.C) \in S \), \( y \) is a fresh variable, and there is no \( z \) s.t. \( (x r z) \in S \) and \( (z : C) \in S \).

4. \( S \rightarrow \forall \{ y : C \} \cup S \)
   if \( (x : \forall r.C), (x r y) \in S \) and \( (y : C) \notin S \).

Notice: Deterministic rules (1,3,4) vs. non-deterministic (2).
Generating rules (3) vs. non-generating (1,2,4).
Theorem (Invariance)

Let $S$ and $T$ be constraint systems.

1. If $T$ has been generated by applying a deterministic rule to $S$, then $S$ is satisfiable if and only if $T$ is satisfiable.

2. If $T$ has been generated by applying a non-deterministic rule to $S$, then $S$ is satisfiable if $T$ is satisfiable. Furthermore, if a non-deterministic rule can be applied to $S$, then it can be applied such that $S$ is satisfiable if and only if the resulting system $T$ is satisfiable.

Theorem (Termination)

Let $C$ be an $\mathcal{ALC}$ concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{x : C\}$. 
A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form \( x : A \) and \( x : \neg A \), where \( A \) is a concept name.

**Theorem (Soundness and Completeness)**

*A closed constraint system is satisfiable if and only it does not contain a clash.*

**Proof idea.**

\( \Rightarrow \): obvious. \( \Leftarrow \): Construct a model by using the concept labels.
Space requirements

Because the tableau method is non-deterministic (→□ rule), there could be exponentially many closed constraint systems in the end.
Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

Example

\[
\exists r. A \sqcap \exists r. B \\
\forall r. ( \exists r. A \sqcap \exists r. B \\
\forall r. ( \exists r. A \sqcap \exists r. B \\
\forall r. (\ldots)))
\]

However: One can modify the algorithm so that it needs only polynomial space.
Idea: Generate a y only for one ∃r.C and then proceed into the depth.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):

- **Normalize** and **unfold** and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in $\mathcal{ALC}$ we do not need this because we are never **forced** to identify two objects.
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