Principles of Knowledge Representation and Reasoning
Seminar Networks and Description Logics III: Description Logics – Reasoning Services and Reductions

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Motivation
Example TBox & ABox

\[
\begin{align*}
\text{Male} & \equiv \neg \text{Female} \\
\text{Human} & \sqsubseteq \text{Living_entity} \\
\text{Woman} & \equiv \text{Human} \sqcap \text{Female} \\
\text{Man} & \equiv \text{Human} \sqcap \neg \text{Male} \\
\text{Mother} & \equiv \text{Woman} \sqcap \exists \text{has-child} \cdot \text{Human} \\
\text{Father} & \equiv \text{Man} \sqcap \exists \text{has-child} \cdot \text{Human} \\
\text{Parent} & \equiv \text{Father} \sqcup \text{Mother} \\
\text{Grandmother} & \equiv \text{Woman} \sqcap \exists \text{has-child} \cdot \text{Parent} \\
\text{Mother-without-daughter} & \equiv \text{Mother} \sqcap \forall \text{has-child} \cdot \text{Male} \\
\text{Mother-with-many-children} & \equiv \text{Mother} \sqcap (\geq 3 \text{has-child}) \\
\end{align*}
\]

DIANA: Woman
ELIZABETH: Woman
CHARLES: Man
EDWARD: Man
ANDREW: Man

DIANA: Mother-without-daughter
(ELIZABETH, CHARLES): has-child
(ELIZABETH, EDWARD): has-child
(ELIZABETH, ANDREW): has-child
(DIANA, WILLIAM): has-child
(CHARLES, WILLIAM): has-child
Motivation: Reasoning services

What do we want to know?

- We want to check whether the knowledge base is reasonable:
  - Is each defined concept in a TBox satisfiable?
  - Is a given TBox satisfiable?
  - Is a given ABox satisfiable?
- What can we conclude from the represented knowledge?
  - Is concept \( X \) subsumed by concept \( Y \)?
  - Is an object a instance of a concept \( X \)?
- These problems can be reduced to logical satisfiability or implication – using the logical semantics.
- However, we take a different route: we will try to simplify these problems and then we specify direct inference methods.
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Basic Reasoning Services
Satisfiability of concept descriptions

Given a concept description $C$ in “isolation”, i.e., in an empty TBox, is $C$ satisfiable?

Test:
- Does there exist an interpretation $\mathcal{I}$ such that $C^\mathcal{I} \neq \emptyset$?
- Translated into FOL: Is the formula $\exists x \, C(x)$ satisfiable?

Example

$\text{Woman} \sqcap (\leq 0 \text{ has-child}) \sqcap (\geq 1 \text{ has-child})$ is unsatisfiable.
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Satisfiability of concept descriptions in a TBox

Given a TBox $\mathcal{T}$ and a concept description $C$, is $C$ satisfiable?

Test:
- Does there exist a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$?
- Translated into FOL: Is the formula $\exists x \ C(x)$ together with the formulae resulting from the translation of $\mathcal{T}$ satisfiable?

Example
Mother-without-daughter $\sqcap \forall$has-child.Female is unsatisfiable, given our previously specified family TBox.
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Eliminating the TBox
Reduction: Getting rid of the TBox

We can reduce satisfiability problem of concept descriptions in a TBox to the satisfiability problem of concept descriptions in the empty TBox.

Idea:

- Since TBoxes are cycle-free, one can understand a concept definition as a kind of “macro”.
- For a given TBox $\mathcal{T}$ and a given concept description $C$, all defined concept symbols appearing in $C$ can be expanded until $C$ contains only undefined concept symbols.
- An expanded concept description is then satisfiable if and only if $C$ is satisfiable in $\mathcal{T}$.
- **Problem**: What do we do with partial definitions (using $\sqsubseteq$)?
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- An expanded concept description is then satisfiable if and only if $C$ is satisfiable in $\mathcal{T}$.
- **Problem**: What do we do with partial definitions (using $\sqsubseteq$)?
A terminology is called normalized when it does not contain definitions of the form $A \sqsubseteq C$.

In order to normalize a terminology, replace $A \sqsubseteq C$ by $A \equiv A^* \sqcap C$, where $A^*$ is a fresh concept symbol (not appearing elsewhere in $T$).

If $T$ is a terminology, the normalized terminology is denoted by $\tilde{T}$. 
Normalizing is reasonable

**Theorem (Normalization invariance)**

If $\mathcal{I}$ is a model of the terminology $\mathcal{T}$, then there exists a model $\mathcal{I}'$ of $\tilde{\mathcal{T}}$ such that for all concept symbols $A$ occurring in $\mathcal{T}$, it holds $A^\mathcal{I} = A^\mathcal{I}'$, and *vice versa*.

**Proof.**

“⇒”: Let $\mathcal{I}$ be a model of $\mathcal{T}$. This model should be extended to $\mathcal{I}'$ so that the freshly introduced concept symbols also get interpretations. Assume $(A \subseteq C) \in \mathcal{T}$, i.e., we have $(A \equiv A^* \cap C) \in \tilde{\mathcal{T}}$. Then set $A^*\mathcal{I}' := A^\mathcal{I}$. $\mathcal{I}'$ obviously satisfies $\tilde{\mathcal{T}}$ and has the same interpretation for all symbols in $\mathcal{T}$.

“⇐”: Given a model $\mathcal{I}'$ of $\tilde{\mathcal{T}}$, its restriction to symbols of $\mathcal{T}$ is the interpretation we look for.
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"⇐": Given a model $\mathcal{I}'$ of $\tilde{\mathcal{T}}$, its restriction to symbols of $\mathcal{T}$ is the interpretation we look for.
We say that a normalized TBox is unfolded by one step when all defined concept symbols on the right sides are replaced by their defining terms.

**Example:** Mother ≜ Woman ⊓ ... is unfolded to Mother ≜ (Human ⊓ Female) ⊓ ...

We write $U(T)$ to denote a one-step unfolding and $U^n(T)$ to denote an $n$-step unfolding.

We say that $T$ is unfolded if $U(T) = T$.

$U^n(T)$ is called the unfolding of $T$ if $U^n(T) = U^{n+1}(T)$. If such an unfolding exists, it is denoted by $\hat{T}$.
TBox unfolding

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Theorem (Existence of unfolded terminology)

Each normalized terminology $\mathcal{T}$ can be unfolded, i.e., its unfolding $\hat{\mathcal{T}}$ exists.

Proof idea.

The main reason is that terminologies have to be cycle-free. The proof can be done by induction of the definition depth of concepts.
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The main reason is that terminologies have to be cycle-free. The proof can be done by induction of the definition depth of concepts.
Theorem (Model equivalence for unfolded terminologies)

\[ \mathcal{I} \] is a model of a normalized terminology \( \mathcal{T} \) if and only if it is a model of \( \hat{T} \).

Proof sketch.

\( \Rightarrow \): Let \( \mathcal{I} \) be a model of \( \mathcal{T} \). Then it is also a model of \( U(\mathcal{T}) \), since on the right side of the definitions only terms with identical interpretations are substituted. However, then it must also be a model of \( \hat{T} \).

\( \Leftarrow \): Let \( \mathcal{I} \) be a model for \( U(\mathcal{T}) \). Clearly, this is also a model of \( \mathcal{T} \) (with the same argument as above). This means that any model \( \hat{T} \) is also a model of \( \mathcal{T} \).
Properties of unfoldings (2): Equivalence

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**Theorem (Model equivalence for unfolded terminologies)**

$I$ is a model of a normalized terminology $\mathcal{T}$ if and only if it is a model of $\hat{\mathcal{T}}$.

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$\implies$: Let $I$ be a model of $\mathcal{T}$. Then it is also a model of $U(\mathcal{T})$, since on the right side of the definitions only terms with identical interpretations are substituted. However, then it must also be a model of $\hat{\mathcal{T}}$.

$\impliedby$: Let $I$ be a model for $U(\mathcal{T})$. Clearly, this is also a model of $\mathcal{T}$ (with the same argument as above). This means that any model $\hat{\mathcal{T}}$ is also a model of $\mathcal{T}$. 

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**Theorem (Model equivalence for unfolded terminologies)**

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Properties of unfoldings (2): Equivalence

**Theorem (Model equivalence for unfolded terminologies)**

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$\Leftarrow$: Let $I$ be a model for $U(T)$. Clearly, this is also a model of $T$ (with the same argument as above). This means that any model $\hat{T}$ is also a model of $T$. □
Generating models

- All concept and role names not occurring on the left hand side of definitions in a terminology $\mathcal{T}$ are called \textit{primitive components}.
- Interpretations restricted to primitive components are called \textit{initial interpretations}.

**Theorem (Model extension)**

For each initial interpretation $\mathcal{J}$ of a normalized TBox, there exists a unique interpretation $\mathcal{I}$ extending $\mathcal{J}$ and satisfying $\mathcal{T}$.

**Proof idea.**

Use $\widehat{T}$ and compute an interpretation for all defined symbols.

**Corollary (Model existence for TBoxes)**

Each TBox has at least one model.
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For each initial interpretation $\mathcal{I}$ of a normalized TBox, there exists a unique interpretation $\mathcal{I}_{\text{ext}}$ extending $\mathcal{I}$ and satisfying $\mathcal{T}$.

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Unfolding of concept descriptions

- Similar to the unfolding of TBoxes, we can define the unfolding of a concept description.
- We write $\hat{C}$ for the unfolded version of $C$.

Theorem (Satisfiability of unfolded concepts)

An concept description $C$ is satisfiable in a terminology $\mathcal{T}$ if and only if $\hat{C}$ satisfiable in an empty terminology.

Proof.

“$\Rightarrow$”: trivial.

“$\Leftarrow$”: Use the interpretation for all the symbols in $\hat{C}$ to generate an initial interpretation of $\mathcal{T}$. Then extend it to a full model $\mathcal{I}$ of $\mathcal{T}$. This satisfies $\mathcal{T}$ as well as $\hat{C}$. Since $\hat{C}^\mathcal{I} = C^\mathcal{I}$, it satisfies also $C$.  

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General TBox Reasoning Services
Subsumption in a TBox

Given a terminology $\mathcal{T}$ and two concept descriptions $C$ and $D$, is $C$ subsumed by (or a sub-concept of) $D$ in $\mathcal{T}$ (symb. $C \sqsubseteq_{\mathcal{T}} D$)?

Test:

- Is $C$ interpreted as a subset of $D$ in each model $\mathcal{I}$ of $\mathcal{T}$, i.e. $C^\mathcal{I} \subseteq D^\mathcal{I}$?
- Is the formula $\forall x (C(x) \rightarrow D(x))$ a logical consequence of the translation of $\mathcal{T}$ into FOL?

Example

Given our family TBox, it holds Grandmother $\sqsubseteq_{\mathcal{T}}$ Mother.
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Example

Given our family TBox, it holds Grandmother $\sqsubseteq_{\mathcal{T}}$ Mother.
Subsumption (without a TBox)

Given two concept descriptions $C$ and $D$, is $C$ subsumed by $D$ regardless of a TBox (or in an empty TBox) (symb. $C \sqsubseteq D$)?

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- Is $C$ interpreted as a subset of $D$ for all interpretations $\mathcal{I}$ ($C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$)?
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Example

Clearly, Human $\sqcap$ Female $\sqsubseteq$ Human.
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Clearly, Human $\cap$ Female $\sqsubseteq$ Human.
Subsumption in a TBox can be reduced to subsumption in the empty TBox:

\[ \ldots \text{ normalize and unfold TBox and concept descriptions.} \]

Subsumption in the empty TBox can be reduced to unsatisfiability:

\[ \ldots C \sqsubseteq D \text{ iff } C \sqcap \neg D \text{ is unsatisfiable.} \]

Unsatisfiability can be reduced to subsumption:

\[ \ldots C \text{ is unsatisfiable iff } C \sqsubseteq (C \sqcap \neg C). \]
Subsumption in a TBox can be reduced to subsumption in the empty TBox:
\[ \ldots \text{ normalize and unfold TBox and concept descriptions.} \]

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Unsatisfiability can be reduced to subsumption:
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Reductions

- Subsumption in a TBox can be reduced to subsumption in the empty TBox:
  
  ... normalize and unfold TBox and concept descriptions.

- Subsumption in the empty TBox can be reduced to unsatisfiability:
  
  ... $C \sqsubseteq D$ iff $C \cap \neg D$ is unsatisfiable.

- Unsatisfiability can be reduced to subsumption:
  
  ... $C$ is unsatisfiable iff $C \sqsubseteq (C \cap \neg C)$. 
Classification

Compute all subsumption relationships (and represent them using only a minimal number of relationships)!

Useful in order to:

- check the modeling
- use the precomputed relations later when subsumption queries have to be answered

Problem can be reduced to subsumption checking: then it is a generalized sorting problem!
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Example
General ABox Reasoning Services
Satisfiability of an ABox

Given an ABox $\mathcal{A}$, does this set of assertions have a model?

- **Notice**: ABoxes representing the real world, should always have a model.

Example

The ABox

- $X : (\forall r. \neg C)$,
- $Y : C$,
- $(X, Y) : r$

is not satisfiable.
ABox satisfiability

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ABox satisfiability in a TBox

Given an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, is $\mathcal{A}$ consistent with the terminology introduced in $\mathcal{T}$, i.e., is $\mathcal{T} \cup \mathcal{A}$ satisfiable?

Example

If we extend our example with

- MARGRET: Woman
- (DIANA,MARGRET): has-child,

then the ABox becomes unsatisfiable in the given TBox.

- Problem is reducible to satisfiability of an ABox:
  - … normalize terminology, then unfold all concept and role descriptions in the ABox
ABox satisfiability in a TBox

Given an ABox \( A \) and a TBox \( T \), is \( A \) consistent with the terminology introduced in \( T \), i.e., is \( T \cup A \) satisfiable?

Example

If we extend our example with

\[
\text{MARGRET: Woman} \\
\text{(DIANA,MARGRET): has-child,}
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Instance relations

Which additional ABox formulae of the form \( a : C \) follow logically from a given ABox and TBox?

- Is \( a^\mathcal{I} \in C^\mathcal{I} \) true in all models \( \mathcal{I} \) of \( T \cup A \)?
- Does the formula \( C(a) \) logically follow from the translation of \( A \) and \( T \) to predicate logic?

Reductions:

- Instance relations wrt. an ABox and a TBox can be reduced to instance relations wrt. ABox: use normalization and unfolding
- Instance relations in an ABox can be reduced to ABox unsatisfiability:

\[
 a : C \text{ holds in } A \iff A \cup \{a : \neg C\} \text{ is unsatisfiable}
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**Reductions:**

- Instance relations wrt. an ABox and a TBox can be reduced to instance relations wrt. ABox: use **normalization** and **unfolding**

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Example

ELIZABETH: Mother-with-many-children?

WILLIAM: ¬ Female

ELIZABETH: Mother-without-daughter?

no (no CWA!)

ELIZABETH: Grandmother?

no (only male, but not necessarily human!)
Examples

Example

- ELIZABETH: Mother-with-many-children?
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Examples

Example

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**Example**

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Realization

For a given object \( a \), determine the \textbf{most specialized concept symbols} such that \( a \) is an instance of these concepts.

**Motivation:**
- Similar to \textbf{classification}
- Is the minimal representation of the instance relations (in the set of concept symbols)
- Will give us faster answers for instance queries!

**Reduction:** Can be reduced to (a sequence of) instance relation tests.
Realization

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**Reduction:** Can be reduced to (a sequence of) instance relation tests.
Given a concept description $C$, determine the set of all (specified) instances of the concept description.

**Example**

We ask for all instances of the concept `Male`. For our TBox/ABox we will get the answer CHARLES, ANDREW, EDWARD, WILLIAM.

- **Reduction**: Compute the set of instances by testing the instance relation for each object!
- **Implementation**: Realization can be used to speed this up
Given a concept description $C$, determine the set of all (specified) instances of the concept description.

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We ask for all instances of the concept Male. For our TBOX/ABox we will get the answer CHARLES, ANDREW, EDWARD, WILLIAM.

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- **Implementation**: Realization can be used to speed this up
Summary and Outlook
Reasoning services – summary

- Satisfiability of concept descriptions
  - in a given TBox or in an empty TBox
- Subsumption between concept descriptions
  - in a given TBox or in an empty TBox
- Classification
- Satisfiability of an ABox
  - in a given TBox or in an empty TBox
- Instance relations in an ABox
  - in a given TBox or in an empty TBox
- Realization
- Retrieval
Outlook

- How to determine **subsumption** between two concept descriptions (in the empty TBox)?
- How to determine **instance relations/ABox satisfiability**?
- How to implement the mentioned reductions **efficiently**?
- Does normalization and unfolding introduce another source of **computational complexity**?