Allen’s Interval Calculus – Outline

1 Allen’s Interval Calculus
   - Motivation
   - Intervals and Relations Between Them
   - Composing Interval Relations

2 Reasoning in Allen’s Interval Calculus

3 A Maximal Tractable Sub-Algebra

4 Literature
Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- **NLP** – we do not have precise time points
- **Planning** – we do not want to commit to time points too early
- **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- **Time intervals**: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?
Consider a planning scenario for multimedia generation:

- P1: Display Picture1
- P2: Say “Put the plug in.”
- P3: Say “The device should be shut off.”
- P4: Point to Plug-in-Picture1.

Temporal relations between events:

- P2 should happen during P1
- P3 should happen during P1
- P2 should happen before or directly precede P3
- P4 should happen during or end together with P2

⇝ P4 happens before or directly precedes P3
⇝ We could add the statement “P4 does not overlap with P3” without creating an inconsistency.
Allen’s Interval Calculus

- Allen’s interval calculus: **time intervals** and **binary relations** over them

- **Time intervals**: $X = (X^-, X^+)$, where $X^-$ and $X^+$ are interpreted over the reals and $X^- < X^+$ (⇝ naïve approach)

- **Relations** between concrete intervals, e.g.:
  
  - $(1.0, 2.0)$ strictly before $(3.0, 5.5)$
  - $(1.0, 3.0)$ meets $(3.0, 5.5)$
  - $(1.0, 4.0)$ overlaps $(3.0, 5.5)$

  ... 

Which relations are conceivable?
The Base Relations

How many ways are there to order the four points of two intervals?

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Name</th>
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<tbody>
<tr>
<td>((X, Y) : X^- &lt; X^+ &lt; Y^- &lt; Y^+)</td>
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<td>((X, Y) : Y^- &lt; X^- &lt; X^+ = Y^+)</td>
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<td>((X, Y) : Y^- &lt; X^- &lt; X^+ &lt; Y^+)</td>
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<td>((X, Y) : Y^- = X^- &lt; X^+ = Y^+)</td>
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and the **converse** relations (obtained by exchanging \(X\) and \(Y\))

\(\leadsto\) These relations are JEPD.
The 13 Base Relations Graphically

before

meets

overlaps

during

starts

finishes

equals

before\(^{-1}\)

meets\(^{-1}\)

overlaps\(^{-1}\)

during\(^{-1}\)

starts\(^{-1}\)

finishes\(^{-1}\)
Assumption: We don’t have precise information about the relation between $X$ and $Y$, e.g.:

$$X \circ Y \text{ or } X \mathbin{m} Y$$

...modelled by sets of base relations (meaning the union of the relations):

$$X \{\circ, m\} Y$$

$\sim \sim 2^{13}$ imprecise relations (incl. $\emptyset$ and $B$)

Example of an indefinite qualitative description:

$$\left\{ X \{\circ, m\} Y, Y \{m\} Z, X \{\circ, m\} Z \right\}$$
Our Example... Formally

P1: Display Picture1
P3: Say “The device should be shut off.”

P2: Say “Put the plug in.”
P4: Point to Plug-in-Picture1.

Compose the constraints: $P4 \{d, f\} P2$ and $P2 \{d\} P1$}

$\Rightarrow P4 \{d\} P1$. 
## Composition of Base Relations

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Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.

What would be a systematic approach?

How costly is that?

Is that complete?

If not, could it be complete on a subset of the relation system?
Reasoning in Allen’s Interval Calculus

1 Allen’s Interval Calculus

2 Reasoning in Allen’s Interval Calculus
   - Enforcing Path Consistency
   - NP-Hardness Example
   - The Continuous Endpoint Class
   - Completeness for the CEP Class

3 A Maximal Tractable Sub-Algebra

4 Literature
Constraint Propagation – The Naive Algorithm

Enforcing path consistency using the straight-forward method:
Let $Table[i,j]$ be an array of size $n \times n$ ($n$: number of intervals) in which we record the constraints between the intervals.

**EnforcePathConsistency1($C$)**

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

repeat
  for each pair $(i,j)$, $1 \leq i,j \leq n$
    for each $k$ with $1 \leq k \leq n$
      $Table[i,j] := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
  until no entry in $Table$ is changed

\(\rightsquigarrow\) terminates;

\(\rightsquigarrow\) needs $O(n^5)$ intersections and compositions.

An $O(n^3)$ Algorithm

EnforcePathConsistency2($C$)

**Input:** a (binary) CSP $C = \langle V, D, C \rangle$

**Output:** an equivalent, but path consistent CSP $C'$

$Paths(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}$

Queue := $\bigcup_{i,j} Paths(i,j)$

**while** Queue $\neq \emptyset$

select and delete $(i,k,j)$ from Queue

$T := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$

**if** $T \neq Table[i,j]$

$Table[i,j] := T$

$Table[j,i] := T^{-1}$

Queue := Queue $\cup$ Paths($i,j$)
Example for Incompleteness

```
Example for Incompleteness

\[ \{s, m\} \rightarrow \{s, m\} \]

\[ \{d, d^{-1}\} \rightarrow \{o\} \rightarrow \{d, d^{-1}\} \]

\[ \{f, f^{-1}\} \]

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NP-Hardness

Theorem (Kautz & Vilain)

*CSAT is NP-hard for Allen’s interval calculus.*

**Beweis.**

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

\[
\begin{align*}
1 & \quad \{m\} & 2 \\
2 & \quad \{m\} & 3 \\
v_i & \quad \{m, \equiv, m^{-1}\} & 2 & \forall v_i \in V \\
v_i & \quad \{m, m^{-1}, <, >\} & v_j & \forall (v_i, v_j) \in E
\end{align*}
\]

This constraint system is satisfiable iff $G$ can be colored with 3 colors.
Looking for Special Cases

- **Idea**: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.

- **Note**: Interval formulae $X R Y$ can be expressed as clauses over atoms of the form $a \, op \, b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $op \in \{<, >, =, \leq, \geq\}$.

- **Example**: All base relations can be expressed as unit clauses.

**Lemma**

Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.
The Continuous Endpoint Class

**Continuous Endpoint Class** $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

**Example:** All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^+\}$$

![Diagram of X and Y intervals](image-url)
Why Do We Have Completeness?

The set $\mathcal{C}$ is **closed** under intersection, composition, and converse (it is a **sub-algebra** wrt. these three operations on relations). This can be shown by using a computer program.

**Lemma**

*Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.*

**Theorem (van Beek)**

*Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.*

(Proof: Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal.)

**Corollary**

*A path consistent interval CSP consisting of base relations only is satisfiable.*
Helly’s Theorem

Definition

A set $M \subseteq \mathbb{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

Theorem (Helly)

Let $F$ be a finite family of at least $n + 1$ convex sets in $\mathbb{R}^n$. If all sub-families of $F$ with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$. 
Strong $n$-Consistency (1)

Proof (Part 1).

We prove the claim by induction over $k$ with $k \leq n$.

Base case: $k = 1, 2, 3$  \( \checkmark \)

Induction assumption: Assume strong $(k - 1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k - 1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.
Strong $n$-Consistency (2): Instantiating the $k$th Variable

Proof (Part 2).

The instantiation of the $k - 1$ variables $X_i$ to $(s_i, e_i)$ restricts the instantiation of $X_k$.

Note: Since $R_{ij} \in C$ by assumption, these restrictions can be expressed by inequalities of the form:

\[ s_i < X_k^+ \land e_j \geq X_k^- \land \ldots \]

Such inequalities define convex subsets in $\mathbb{R}^2$.

\[ \rightarrow \text{Consider sets of 3 inequalities (} \approx 3 \text{ convex sets).} \]
Strong $n$-Consistency (3): Using Helly’s Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention $X_k^-$ and $X_k^+$, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by 3-consistency there exists a common point.

$\Rightarrow$ With Helly’s Theorem, there exists an instantiation consistent with all inequalities.

$\Rightarrow$ Strong $k$-consistency for all $k \leq n$. 

$\square$
C\text{MIN}(\mathcal{C}) \text{ can be computed in } O(n^3) \text{ time (for } n \text{ being the number of intervals) using the path consistency algorithm.}

\mathcal{C} \text{ is a set of relations occurring “naturally” when observations are uncertain.}

\mathcal{C} \text{ contains 83 relations (incl. the impossible and the universal relations).}

Are there larger sets such that path consistency computes minimal CSPs? \text{Probably not.}

Are there larger sets of relations that permit polynomial satisfiability testing? \text{Yes.}
1 Allen’s Interval Calculus

2 Reasoning in Allen’s Interval Calculus

3 A Maximal Tractable Sub-Algebra
   ■ The Endpoint Subclass
   ■ The ORD-Horn Subclass
   ■ Maximality
   ■ Solving Arbitrary Allen CSPs

4 Literature
The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(\{d, o\} Y) = \{ X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, X^+ < Y^+ \}$$

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides $\text{CSAT}(\mathcal{P})$.  

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The ORD-Horn Subclass

ORD-Horn Subclass: \( \mathcal{H} \subseteq \mathcal{A} \) is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

\[ a \leq b, a = b, a \neq b \]

\( \neg a \leq b \) is not allowed!

Example: all \( R \in \mathcal{P} \) and \( \{ o, s, f^{-1} \} \):

\[
\pi(X\{o, s, f^{-1}\}Y) = \left\{ X^-, X^- \neq X^+, \right. \\
Y^- \leq Y^+, Y^- \neq Y^+, \\
X^- \leq Y^-, \\
X^- \leq Y^+, X^- \neq Y^+, \\
Y^- \leq X^+, X^+ \neq Y^-, \\
X^+ \leq Y^+, \\
\left. X^- \neq Y^- \lor X^+ \neq Y^+ \right\}.
\]
Partial Orders: The \textit{ORD} Theory

Let \textit{ORD} be the following theory:

\[
\begin{align*}
\forall x, y, z : \quad x \leq y \land y \leq z & \rightarrow x \leq z \quad \text{(transitivity)} \\
\forall x : \quad x \leq x & \quad \text{(reflexivity)} \\
\forall x, y : \quad x \leq y \land y \leq x & \rightarrow x = y \quad \text{(anti-symmetry)} \\
\forall x, y : \quad x = y & \rightarrow x \leq y \quad \text{(weakening of \(=\))} \\
\forall x, y : \quad x = y & \rightarrow y \leq x \quad \text{(weakening of \(=\)).}
\end{align*}
\]

- \textit{ORD} describes partially ordered sets, \(\leq\) being the ordering relation.
- \textit{ORD} is a Horn theory
- What is missing wrt. dense and linear orders?
Satisfiability over Partial Orders

Proposition

Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup \text{ORD}$ is satisfiable over arbitrary interpretations.

Beweis.

$\Rightarrow$: Since the reals form a partially ordered set (i.e., satisfy ORD), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order.

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Complexity of CSAT(\mathcal{H})

Let $ORD_\pi(\Theta)$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

**Proposition**

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_\pi(\Theta) \cup \pi(\Theta)$ is so.

**Proof idea:** Herbrand expansion!

**Theorem**

CSAT(\mathcal{H}) can be decided in polynomial time.

**Beweis.**

CSAT(\mathcal{H}) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time.

$C \subset P \subset H$ with $|C| = 83$, $|P| = 188$, $|H| = 868$
Lemma

Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \iff \Theta \text{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

*Enforcing path consistency decides CSAT($\mathcal{H}$).*

$\Rightarrow$ Maximality of $\mathcal{H}$?

$\Rightarrow$ Do we have to check all $8192 - 868$ extensions?
Complexity of Sub-Algebras

Let \( \hat{S} \) be the closure of \( S \subseteq \mathcal{A} \) under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by \( S \)).

**Theorem**

\( \text{CSAT}(\hat{S}) \) can be polynomially transformed to \( \text{CSAT}(S) \).

**Proof Idea.**

All relations in \( \hat{S} – S \) can be modeled by a fixed number of compositions, intersections, and conversions of relations in \( S \), introducing perhaps some fresh variables.

\( \Rightarrow \) Polynomiality of \( S \) extends to \( \hat{S} \).

\( \Rightarrow \) NP-hardness of \( \hat{S} \) is inherited by all generating sets \( S \).

\( \Rightarrow \) Note: \( \mathcal{H} = \hat{\mathcal{H}} \).
Minimal Extensions of the $\mathcal{H}$-Subclass

A computer-aided case analysis leads to the following result:

**Lemma**

*There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$*

\[
N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1
\]
\[
N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2
\]

The clause form of these relations contain “proper” disjunctions!

**Theorem**

$\text{CSAT}(\mathcal{H} \cup \{N_i\})$ is NP-complete.

**Question:** Are there other maximal tractable subclasses?
Interesting subclasses of $A$ should contain all basic relations.

A computer-aided case analysis reveals:
For $S \supseteq \{ \{ B \} : B \in B \}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. $N_1$ or $N_2$ is in $\hat{S}$.

In case 2, one can show: CSAT($S$) is NP-complete.

$\Rightarrow \mathcal{H}$ is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.
Relevance?

Theory: ⊕ We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

Practice: ⊋ All known applications either need only $P$ or they need more than $H$!

Backtracking methods might profit from the result by reducing the branching factor.

~~ How difficult is CSAT($\mathcal{A}$) in practice?

~~ What are the relevant branching factors?
Solving General Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

Which tractable fragment should one use?

Backtracking algorithm using path consistency as a forward-checking method
Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
Refinements and evaluation of different heuristics
Which tractable fragment should one use?
Branching Factors

- If the labels are split into base relations, then on average a label is split into
  
  **6.5 relations**

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into
  
  **2.955 relations**

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into
  
  **2.533 relations**

~⇒ A difference of **0.422**

~⇒ This makes a difference for “hard” instances.
Allen’s interval calculus is often adequate for describing relative orders of events that have duration.

The satisfiability problem for CSPs using the relations is NP-complete.

For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.

For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.

Can be used in practice for backtracking algorithms.
Literature I

J. F. Allen.
Maintaining knowledge about temporal intervals.
Also in Readings in Knowledge Representation.

P. van Beek and R. Cohen.
Exact and approximate reasoning about temporal relations.

B. Nebel and H.-J. Bürckert.
Reasoning about temporal relations: A maximal tractable subclass of
Allen’s interval algebra.

B. Nebel.
Solving hard qualitative temporal reasoning problems: Evaluating the
efficiency of using the ORD-Horn class.
A complete classification of complexity in Allen’s algebra in the presence of a non-trivial basic relation.

Reasoning about Temporal Relations: The Tractable Subalgebras of Allen’s Interval Algebra.