Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning: Allen's Interval Calculus

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Allen's Interval Calculus

Qualitative temporal representation and reasoning

Often we do not want to talk about precise times:

- NLP – we do not have precise time points
- Planning – we do not want to commit to time points too early
- Scenario descriptions – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- Time points: actions and events are instantaneous, or we consider their beginning and ending
- Time intervals: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?

Motivation: An example

Consider a planning scenario for multimedia generation:

P1: Display Picture1
P2: Say "Put the plug in."
P3: Say "The device should be shut off."
P4: Point to Plug-in-Picture1.

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

- P4 happens before or directly precedes P3
- We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

- Allen's interval calculus: time intervals and binary relations over them
- Time intervals: $X = (X^-, X^+)$, where X^- and X^+ are interpreted over the reals and $X^- < X^+$ ($\leadsto$ naïve approach)
- Relations between concrete intervals, e. g.:

  (1.0,2.0) strictly before (3.0,5.5)
  (1.0,3.0) meets (3.0,5.5)
  (1.0,4.0) overlaps (3.0,5.5)
  . . .

$\leadsto$ Which relations are conceivable?

The base relations

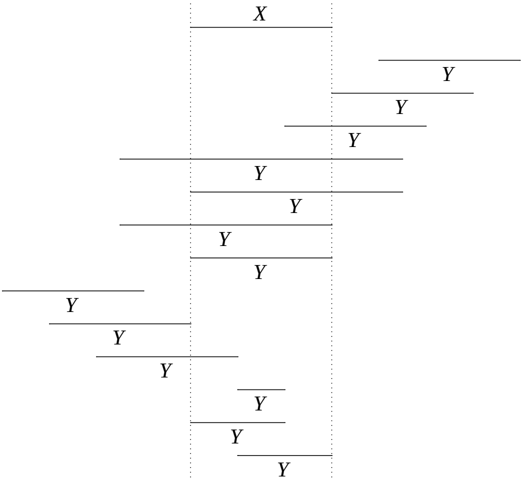
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X,Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X,Y) : X^- < X^+ = Y^- < Y^+\}$	$\mathtt{m}$	meets
$\{(X,Y) : X^- < Y^- < X^+ < Y^+\}$	$\mathtt{o}$	overlaps
$\{(X,Y) : X^- = Y^- < X^+ < Y^+\}$	$\mathtt{s}$	starts
$\{(X,Y) : Y^- < X^- < X^+ = Y^+\}$	$\mathtt{f}$	finishes
$\{(X,Y) : Y^- < X^- < X^+ < Y^+\}$	$\mathtt{d}$	during
$\{(X,Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the **converse** relations (obtained by exchanging X and Y)

$\rightsquigarrow$ These relations are JEPD.

The 13 base relations graphically



before
meets
overlaps
during
starts
finishes
equals
before^{-1}
meets^{-1}
overlaps^{-1}
during^{-1}
starts^{-1}
finishes^{-1}

Disjunctive descriptions

- Assumption: We don't have precise information about the relation between X and Y, e. g.:

$$X \mathtt{o} Y \text{ or } X \mathtt{m} Y$$

- …modelled by sets of base relations (meaning the union of the relations):

$$X \{\mathtt{o},\mathtt{m}\} Y$$

$\rightsquigarrow$ 2^{13} imprecise relations (incl. $\emptyset$ and **B**)

Example of an indefinite qualitative description:

$$\left\{ X \{\mathtt{o},\mathtt{m}\} Y,\ Y \{\mathtt{m}\} Z, X \{\mathtt{o},\mathtt{m}\} Z \right\}$$

Our example … formally

P1: Display Picture1

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P3: Say “The device should be shut off.”

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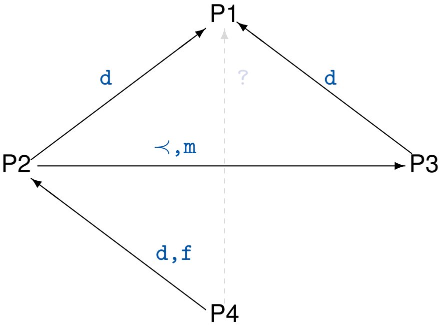

Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$
$\rightsquigarrow P4\,\{\mathtt{d}\}\,P1$.

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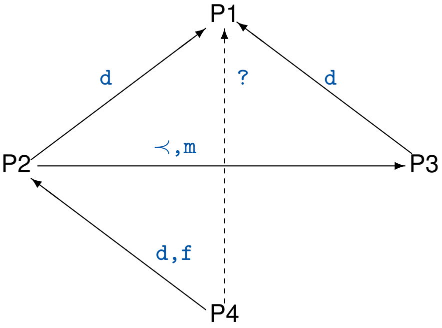

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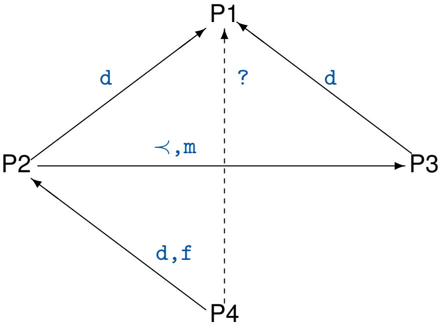

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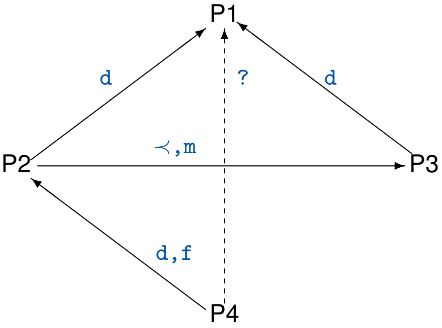


Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$
$\rightsquigarrow$ $P4\,\{\mathtt{d}\}\,P1$.

Composition of base relations

	$\prec$	$\succ$	d	d^{-1}	o	o^{-1}	m	m^{-1}	s	s^{-1}	f	f^{-1}
$\prec$	$\prec$	**B**	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$
$\succ$	**B**	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$	$\succ$
d	$\prec$	$\succ$	d	**B**	$\prec$ o m d s	$\succ$ o^{-1} m^{-1} d f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d f	d	$\prec$ o m d s
d^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	**B**− $\prec\succ$ m m^{-1}	d^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	d^{-1}	o^{-1} d^{-1} s^{-1}	d^{-1}
o	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	**B**− $\prec\succ$ m m^{-1}	$\prec$	o^{-1} d^{-1} s^{-1}	o	d^{-1} f^{-1} o	d s o	$\prec$ o m
o^{-1}	$\prec$ o m, d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$, o^{-1} m^{-1} d^{-1} s^{-1}	**B**− $\prec\succ$ m m^{-1}	$\succ$ o^{-1} m^{-1}	o d^{-1} f^{-1}	$\succ$	o^{-1} d f	o^{-1} $\succ$ m^{-1}	o^{-1}	o^{-1} d^{-1} s^{-1}
m	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$	$\prec$	o d s	$\prec$	f f^{-1} $\equiv$	m	m	d s o	$\prec$
m^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$	o^{-1} d f	$\succ$	s s^{-1} $\equiv$	$\succ$	d f o^{-1}	$\succ$	m^{-1}	m^{-1}
s	$\prec$	$\succ$	d	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	o^{-1} d f	$\prec$	m^{-1}	s	s s^{-1} $\equiv$	d	$\prec$ m o
s^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} .d f	d^{-1}	o d^{-1} f^{-1}	o^{-1}	o d^{-1} f^{-1}	m^{-1}	s s^{-1} $\equiv$	s^{-1}	o^{-1}	d^{-1}
f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\succ$ o^{-1} m^{-1}	m	$\succ$	d	$\succ$ o^{-1} m^{-1}	f	f f^{-1} $\equiv$
f^{-1}	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	d^{-1}	o	o^{-1} d^{-1} s^{-1}	m	s^{-1} o^{-1} d^{-1}	o	d^{-1}	f f^{-1} $\equiv$	f^{-1}

Outlook

- Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.
- What would be a systematic approach?
- How costly is that?
- Is that complete?
- If not, could it be complete on a subset of the relation system?

Reasoning in Allen's Interval Calculus

Enforcing path consistency using the straight-forward method: Let $Table[i,j]$ be an array of size $n \times n$ (n: number of intervals) in which we record the constraints between the intervals.

EnforcePathConsistency1($\mathcal{C}$)

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but path-consistent CSP $\mathcal{C}'$

repeat
  for each pair (i,j), $1 \leq i,j \leq n$
    for each k with $1 \leq k \leq n$
      $Table[i,j] := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
until no entry in *Table* is changed

- $\leadsto$ terminates;
- $\leadsto$ needs $O(n^5)$ intersections and compositions.

An $O(n^3)$ algorithm

EnforcePathConsistency2($\mathcal{C}$)

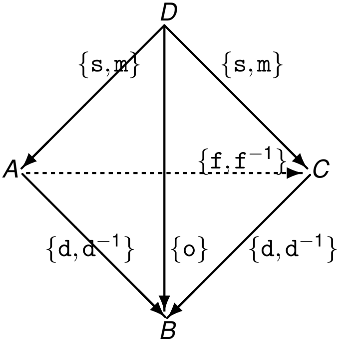
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but path-consistent CSP $\mathcal{C}'$

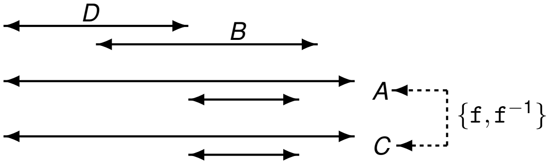
$Paths(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i,j)$

while $Queue \neq \emptyset$
- select and delete (i,k,j) from $Queue$
- $T := Table[i,j] \cap (Table[i,k] \circ Table[k,j])$
- **if** $T \neq Table[i,j]$
  - $Table[i,j] := T$
  - $Table[j,i] := T^{-1}$
  - $Queue := Queue \cup Paths(i,j)$

Example for incompleteness





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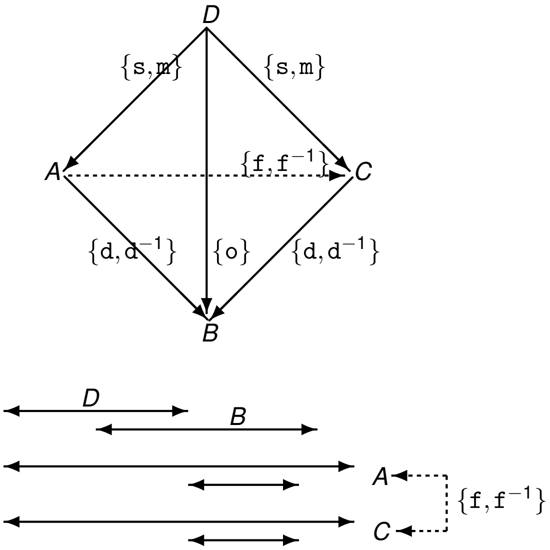

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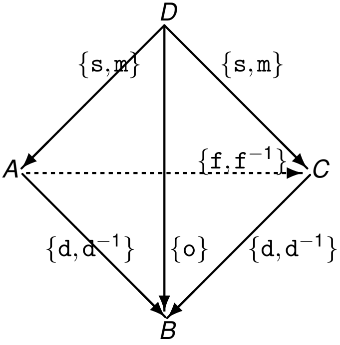

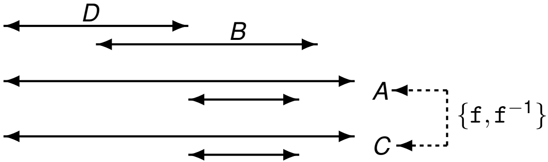

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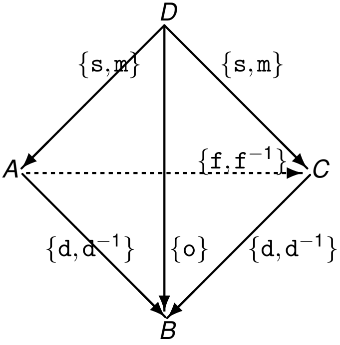
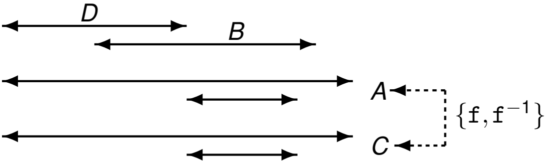




NP-hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \dots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \dots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{cccl} 1 & \{\mathtt{m}\} & 2 & \\ 2 & \{\mathtt{m}\} & 3 & \\ v_i & \{\mathtt{m}, \equiv, \mathtt{m}^{-1}\} & 2 & \forall v_i \in V \\ v_i & \{\mathtt{m}, \mathtt{m}^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable iff G can be colored with 3 colors. □

Looking for special cases

- Idea: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- Note: Interval formulae $X\ R\ Y$ can be expressed as clauses over atoms of the form $a\ op\ b$, where:
  - a and b are endpoints X^-, X^+, Y^- and Y^+ and
  - $op \in \{<, >, =, \leq, \geq\}$.
- Example: All base relations can be expressed as unit clauses.

Lemma

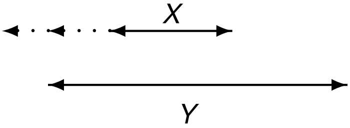
Let $\pi(\Theta)$ be the translation of Θ to clause form. Θ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

The Continuous Endpoint Class

Continuous Endpoint Class $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}$, because

$$\begin{aligned}\pi(X\,\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}\,Y) = \{\, & X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, \\ & X^+ < Y^+\}\end{aligned}$$



Why do we have completeness?

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.

Theorem (van Beek)

Path consistency solves CMIN($\mathcal{C}$) and decides CSAT($\mathcal{C}$).

(Proof: Follows from the above lemma and the fact that a strongly n-consistent CSP is minimal.)

Corollary

A path-consistent interval CSP consisting of base relations only is satisfiable.

Helly's theorem

Definition

A set $M \subseteq \mathbb{R}^n$ is convex if and only if for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M.

Theorem (Helly)

Let F be a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$. If all sub-families of F with $n+1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Strong n-consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3$ $\quad \surd$

Induction assumption: Assume strong $(k-1)$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

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Strong n-consistency (2): Instantiating the kth variable

Proof (Part 2).

The instantiation of the $k-1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \ldots$$

Such inequalities define convex subsets in $\mathbb{R}^2$.

$\rightsquigarrow$ Consider sets of 3 inequalities ($=$ 3 convex sets).

Strong n-consistency (2): Instantiating the kth variable

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Such inequalities define convex subsets in $\mathbb{R}^2$.

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Strong n-consistency (3): Using Helly's theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by 3-consistency there exists a common point.

$\rightsquigarrow$ With Helly's Theorem, there exists an instantiation consistent with all inequalities.

$\rightsquigarrow$ Strong k-consistency for all $k \leq n$. $\square$

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Outlook

- CMIN($\mathcal{C}$) can be computed in $O(n^3)$ time (for n being the number of intervals) using the path consistency algorithm.
- $\mathcal{C}$ is a set of relations occurring "naturally" when observations are uncertain.
- $\mathcal{C}$ contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not.
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.

A Maximal Tractable Sub-Algebra

The EP-subclass

End-Point subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\pi(X\,\{\mathtt{d},\mathtt{o}\}\,Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, X^+ < Y^+\}$$

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides CSAT($\mathcal{P}$).

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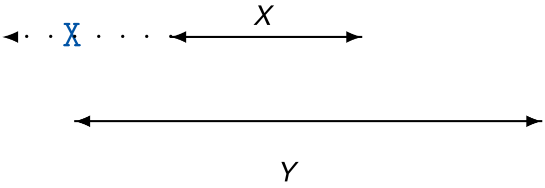


Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

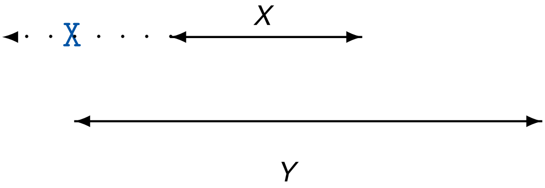
Enforcing path consistency decides CSAT($\mathcal{P}$).

The EP-subclass

End-Point subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\pi(X\,\{\mathtt{d}, \mathtt{o}\}\,Y) = \{\, X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, X^+ < Y^+ \}$$



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The ORD-Horn subclass

ORD-Horn subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{\mathrm{o}, \mathrm{s}, \mathrm{f}^{-1}\}$:

$$\pi(X\{\mathrm{o},\mathrm{s},\mathrm{f}^{-1}\}Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

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Partial orders: The *ORD*-theory

Let *ORD* be the following theory:

$$
\begin{array}{llcll}
\forall x,y,z: & x \leq y \wedge y \leq z & \rightarrow & x \leq z & \text{(transitivity)} \\
\forall x: & x \leq x & & & \text{(reflexivity)} \\
\forall x,y: & x \leq y \wedge y \leq x & \rightarrow & x = y & \text{(anti-symmetry)} \\
\forall x,y: & x = y & \rightarrow & x \leq y & \text{(weakening of } =) \\
\forall x,y: & x = y & \rightarrow & y \leq x & \text{(weakening of } =).
\end{array}
$$

- *ORD* describes partially ordered sets, $\leq$ being the ordering relation.
- *ORD* is a Horn theory
- What is missing wrt. dense and linear orders?

Proposition

Let Θ be a CSP over $\mathcal{H}$. Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

Satisfiability over partial orders

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$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. $\square$

Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Theorem

CSAT($\mathcal{H}$) can be decided in polynomial time.

Proof.

CSAT($\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. □

$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H}$ with $|\mathcal{C}| = 83$, $|\mathcal{P}| = 188$, $|\mathcal{H}| = 868$

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Lemma

Let Θ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides CSAT($\mathcal{H}$).

- $\rightsquigarrow$ Maximality of $\mathcal{H}$?
- $\rightsquigarrow$ Do we have to check all $8192 - 868$ extensions?

Complexity of sub-algebras

Let $\hat{S}$ be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by S).

Theorem

CSAT($\hat{S}$) can be polynomially transformed to CSAT(S).

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in S, introducing perhaps some fresh variables. □

- ⇝ Polynomiality of S extends to $\hat{S}$.
- ⇝ NP-hardness of $\hat{S}$ is inherited by all generating sets S.
- ⇝ Note: $\mathcal{H} = \hat{\mathcal{H}}$.

Minimal extensions of the $\mathcal{H}$-subclass

A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras, $\mathcal{X}_1$ and $\mathcal{X}_2$, that strictly contain $\mathcal{H}$:

$$N_1 = \{\mathtt{d}, \mathtt{d}^{-1}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}\} \in \mathcal{X}_1$$
$$N_2 = \{\mathtt{d}^{-1}, \mathtt{o}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

Theorem

CSAT$(\mathcal{H} \cup \{N_i\})$ is NP-complete.

Question: Are there other maximal tractable subclasses?

"Interesting" subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations.

A computer-aided case analysis reveals:
For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. N_1 or N_2 is in $\hat{S}$.

In case 2, one can show: CSAT(S) is NP-complete.

$\rightsquigarrow$ $\mathcal{H}$ is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.

Theory: $\oplus$ We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

Practice: $\ominus$ All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!

Backtracking methods might profit from the result by reducing the branching factor.

- $\rightsquigarrow$ How difficult is CSAT($\mathcal{A}$) in practice?
- $\rightsquigarrow$ What are the relevant branching factors?

Solving general Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

⇝ Which tractable fragment should one use?

Branching factors

- If the labels are split into base relations, then on average a label is split into

6.5 relations

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into

2.955 relations

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into

2.533 relations

$\rightsquigarrow$ A difference of 0.422

$\rightsquigarrow$ This makes a difference for “hard” instances.

Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

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