Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning: Allen's Interval Calculus

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February 13, 2012

1 Allen’s Interval Calculus

Motivation
Intervals and Relations Between Them
Composing Interval Relations

Qualitative temporal representation and reasoning

Often we do not want to talk about precise times:

- NLP – we do not have precise time points
- Planning – we do not want to commit to time points too early
- Scenario descriptions – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- Time points: actions and events are instantaneous, or we consider their beginning and ending
- Time intervals: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?
Motivation: An example

Consider a planning scenario for multimedia generation:

P1: Display Picture1
P2: Say “Put the plug in.”
P3: Say “The device should be shut off.”
P4: Point to Plug-in-Picture1.

Temporal relations between events:

- P2 should happen during P1
- P3 should happen during P1
- P2 should happen before or directly precede P3
- P4 should happen during or end together with P2
- P4 happens before or directly precedes P3
- We could add the statement “P4 does not overlap with P3” without creating an inconsistency.

The base relations

How many ways are there to order the four points of two intervals?

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Name</th>
</tr>
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<tbody>
<tr>
<td>((X, Y) : X^- &lt; X^+ &lt; Y^- &lt; Y^+)</td>
<td>&lt;</td>
<td>before</td>
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<td>((X, Y) : X^- &lt; X^+ = Y^- &lt; Y^+)</td>
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<td>((X, Y) : X^- = Y^- &lt; X^+ &lt; Y^+)</td>
<td>o</td>
<td>overlaps</td>
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<td>((X, Y) : Y^- &lt; X^- &lt; X^+ = Y^+)</td>
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<td>starts</td>
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<td>((X, Y) : Y^- &lt; X^- &lt; X^+ &lt; Y^+)</td>
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<td>((X, Y) : Y^- = X^- = X^+ &lt; Y^+)</td>
<td>d</td>
<td>during</td>
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<tr>
<td>((X, Y) : Y^- = X^- &lt; X^+ = Y^+)</td>
<td>≡</td>
<td>equal</td>
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and the converse relations (obtained by exchanging X and Y)

~~ These relations are JEPD.

Allen’s Interval Calculus

- Allen’s interval calculus: time intervals and binary relations over them
- Time intervals: \(X = (X^-, X^+)\), where \(X^-\) and \(X^+\) are interpreted over the reals and \(X^- < X^+\) (⇝ naive approach)
- Relations between concrete intervals, e.g.:
  - \((1.0, 2.0)\) strictly before \((3.0, 5.5)\)
  - \((1.0, 3.0)\) meets \((3.0, 5.5)\)
  - \((1.0, 4.0)\) overlaps \((3.0, 5.5)\)

~~ Which relations are conceivable?
Disjunctive descriptions

- Assumption: We don’t have precise information about the relation between $X$ and $Y$, e.g.: $X \circ Y$ or $X \cdot Y$
- ...modelled by sets of base relations (meaning the union of the relations): $X \{o,m\} Y$
- $\sim 2^{13}$ imprecise relations (incl. $\emptyset$ and $B$)

Example of an indefinite qualitative description:

$$\{ X \{o,m\} Y, \ Y \{m\} Z, X \{o,m\} Z \}$$

Our example … formally

P1: Display Picture1
P2: Say “Put the plug in.”
P3: Say “The device should be shut off.”
P4: Point to Plug-in-Picture1.

Composition of base relations

<table>
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<tr>
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Outlook

- Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.
- What would be a systematic approach?
- How costly is that?
- Is that complete?
- If not, could it be complete on a subset of the relation system?
2 Reasoning in Allen’s Interval Calculus

- Enforcing Path Consistency
- NP-Hardness Example
- The Continuous Endpoint Class
- Completeness for the CEP Class

Constraint propagation – The naive algorithm

Enforcing path consistency using the straight-forward method:
Let \( \text{Table}[i,j] \) be an array of size \( n \times n \) (\( n \): number of intervals) in which we record the constraints between the intervals.

\[
\text{EnforcePathConsistency1}(C)
\]

\[\text{Input: a (binary) CSP } C = \langle V, D, C \rangle\]

\[\text{Output: an equivalent, but path-consistent CSP } C'\]

\[
\text{repeat}
\]

\[\text{for each pair } (i,j), 1 \leq i,j \leq n\]

\[\text{for each } k \text{ with } 1 \leq k \leq n\]

\[\text{Table}[i,j] := \text{Table}[i,j] \cap (\text{Table}[i,k] \circ \text{Table}[k,j])\]

\[\text{until } \text{no entry in Table is changed}\]

\[\Rightarrow \text{terminates};\]

\[\Rightarrow \text{needs } O(n^5) \text{ intersections and compositions.}\]

An \( O(n^3) \) algorithm

\[
\text{EnforcePathConsistency2}(C)
\]

\[\text{Input: a (binary) CSP } C = \langle V, D, C \rangle\]

\[\text{Output: an equivalent, but path-consistent CSP } C'\]

\[\text{Paths}(i,j) = \{(i,j,k) : 1 \leq k \leq n\} \cup \{(k,i,j) : 1 \leq k \leq n\}\]

\[\text{Queue} := \bigcup_{i,j} \text{Paths}(i,j)\]

\[\text{while } \text{Queue} \neq \emptyset\]

\[\text{select and delete } (i,k,j) \text{ from Queue}\]

\[T := \text{Table}[i,j] \cap (\text{Table}[i,k] \circ \text{Table}[k,j])\]

\[\text{if } T \neq \text{Table}[i,j]\]

\[\text{Table}[i,j] := T\]

\[\text{Table}[i,j] := T^{-1}\]

\[\text{Queue} := \text{Queue} \cup \text{Paths}(i,j)\]

Example for incompleteness
THEOREM (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.
Reduction from 3-colorability (original proof using 3Sat).
Let $G = (V, E), V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability.
Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

1. $\{m\}$ 2  
2. $\{m\}$ 3  
$v_i \{m, m^{-1}\} 2 \; \forall v_i \in V$  
$v_i \{m, m^{-1}, <, >\} \; v_j \; \forall (v_i, v_j) \in E$

This constraint system is satisfiable iff $G$ can be colored with 3 colors.

Why do we have completeness?

The set $C$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over $C$ is globally consistent.

Theorem (van Beek)

Path consistency solves $\text{CMIN}(C)$ and decides $\text{CSAT}(C)$.

(Proof: Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal.)

Corollary

A path-consistent interval CSP consisting of base relations only is satisfiable.
**Helly’s theorem**

**Definition**
A set $M \subseteq \mathbb{R}^n$ is convex if and only if for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

**Theorem (Helly)**
Let $F$ be a finite family of at least $n + 1$ convex sets in $\mathbb{R}^n$. If all sub-families of $F$ with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

**Strong $n$-consistency (1)**

**Proof (Part 1).**
We prove the claim by induction over $k$ with $k \leq n$.

- **Base case:** $k = 1, 2, 3$  \(\checkmark\)
- **Induction assumption:** Assume strong $(k - 1)$-consistency (and non-emptiness of all relations)
- **Induction step:** From the assumption, it follows that there is an instantiation of $k - 1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k - 1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.

**Strong $n$-consistency (2): Instantiating the $k$th variable**

**Proof (Part 2).**
The instantiation of the $k - 1$ variables $X_i$ to $(s_i, e_i)$ restricts the instantiation of $X_k$.

**Note:** Since $R_{ij} \in C$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \land e_j \geq X_k^- \land \ldots$$

Such inequalities define convex subsets in $\mathbb{R}^2$.

\(\Rightarrow\) Consider sets of 3 inequalities (= 3 convex sets).

**Strong $n$-consistency (3): Using Helly’s theorem**

**Proof (Part 3).**

- **Case 1:** All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest).

Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

- **Case 2:** The inequalities mention $X_k^-$ and $X_k^+$, but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

- **Case 3:** The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by 3-consistency there exists a common point.

\(\Rightarrow\) With Helly’s Theorem, there exists an instantiation consistent with all inequalities.

\(\Rightarrow\) Strong $k$-consistency for all $k \leq n$. \(\square\)
Outlook

- CMIN(\(C\)) can be computed in \(O(n^3)\) time (for \(n\) being the number of intervals) using the path consistency algorithm.
- \(C\) is a set of relations occurring “naturally” when observations are uncertain.
- \(C\) contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not.
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.

The EP-subclass

End-Point subclass: \(P \subseteq A\) is the subclass that permits a clause form containing only unit clauses (\(a \neq b\) is allowed).

Example: all basic relations and \(\{d, o\}\) since

\[\pi(X\{d, o\} Y) = \begin{cases} X^- < X^+, Y^- < Y^+, \\
X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\
X^+ < Y^+ \end{cases}\]

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides CSAT(\(P\)).

The ORD-Horn subclass

ORD-Horn subclass: \(H \subseteq A\) is the subclass that permits a clause form containing only Horn clauses where only the following literals are allowed:

\[a \leq b, a = b, a \neq b\]

\(-a \leq b\) is not allowed!

Example: all \(R \in P\) and \(\{o, s, f^{-1}\}\):

\[\pi(X\{o, s, f^{-1}\} Y) = \begin{cases} X^- \leq X^+, X^- \neq X^+, \\
Y^- \leq Y^+, Y^- \neq Y^+,
X^- \leq Y^-,
X^- \leq Y^+, X^- \neq Y^+,
Y^- \leq X^+, X^- \neq Y^-,
X^+ \leq Y^+, \\
X^- \neq Y^- \lor X^+ \neq Y^+ \end{cases}\]
Partial orders: The \textit{ORD}-theory

Let \textit{ORD} be the following theory:

\begin{align*}
\forall x, y, z : & \quad x \leq y \land y \leq z \rightarrow x \leq z \quad \text{(transitivity)} \\
\forall x : & \quad x \leq x \quad \text{(reflexivity)} \\
\forall x, y : & \quad x \leq y \land y \leq x \rightarrow x = y \quad \text{(anti-symmetry)} \\
\forall x, y : & \quad x = y \rightarrow x \leq y \quad \text{(weakening of =)} \\
\forall x, y : & \quad x = y \rightarrow y \leq x \quad \text{(weakening of =)}.
\end{align*}

\textit{ORD} describes partially ordered sets, \(\leq\) being the ordering relation.

\textit{ORD} is a Horn theory

What is missing wrt. dense and linear orders?

Satisfiability over partial orders

Proposition

Let \(\Theta\) be a CSP over \(\mathcal{H}\). \(\Theta\) is satisfiable over interval interpretations if \(\pi(\Theta) \cup \text{ORD}\) is satisfiable over arbitrary interpretations.

Proof.

\(\Rightarrow\): Since the reals form a partially ordered set (i.e., satisfy \textit{ORD}), this direction is trivial.

\(\Leftarrow\): Each extension of a partial order to a linear order satisfies all formulae of the form \(a \leq b\), \(a = b\), and \(a \neq b\) which have been satisfied over the original partial order.

Complexity of \text{CSAT}(\mathcal{H})

Let \(\text{ORD}_{\pi(\Theta)}\) be the propositional theory resulting from instantiating all axioms with the endpoints occurring in \(\pi(\Theta)\).

Proposition

\(\text{ORD} \cup \pi(\Theta)\) is satisfiable iff \(\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)\) is so.

Proof idea: Herbrand expansion!

Theorem

\text{CSAT}(\mathcal{H}) can be decided in polynomial time.

Proof.

\text{CSAT}(\mathcal{H}) instances can be translated into a propositional Horn theory with blowup \(O(n^3)\) according to the previous Prop., and such a theory is decidable in polynomial time.

\[ C \subset P \subset \mathcal{H} \quad \text{with} \quad |C| = 83, \quad |P| = 188, \quad |\mathcal{H}| = 868 \]

Path consistency and the \textit{OH}-class

Lemma

\(X \nsubseteq Y \notin \Theta\) iff \(\Theta\) is satisfiable

Proof idea: One can show that \(\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)\) is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides \text{CSAT}(\mathcal{H}).

\(\Rightarrow\) Maximaliy of \(\mathcal{H}\)?

\(\Rightarrow\) Do we have to check all \(8192 - 868\) extensions?
Complexity of sub-algebras

Let \( \hat{S} \) be the closure of \( S \subseteq A \) under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by \( S \)).

**Theorem**

CSAT(\( \hat{S} \)) can be polynomially transformed to CSAT(S).

**Proof Idea.**

All relations in \( \hat{S} - S \) can be modeled by a fixed number of compositions, intersections, and conversions of relations in \( S \), introducing perhaps some fresh variables.

\( \Rightarrow \) Polynomiaality of \( S \) extends to \( \hat{S} \).

\( \Rightarrow \) NP-hardness of \( \hat{S} \) is inherited by all generating sets \( S \).

\( \Rightarrow \) Note: \( H = \hat{H} \).

Minimal extensions of the \( H \)-subclass

**A computer-aided** case analysis leads to the following result:

**Lemma**

There are only two minimal sub-algebras, \( \lambda_1 \) and \( \lambda_2 \), that strictly contain \( H \):

\[ N_1 = \{ d, d^{-1}, o^{-1}, s^{-1}, f \} \in \lambda_1 \]

\[ N_2 = \{ d^{-1}, o, o^{-1}, s^{-1}, f^{-1} \} \in \lambda_2 \]

The clause form of these relations contain “proper” disjunctions!

**Theorem**

CSAT(\( H \cup \{ N_i \} \)) is NP-complete.

**Question:** Are there other maximal tractable subclasses?

“Interesting” subclasses

**Interesting subclasses** of \( A \) should contain all basic relations.

A computer-aided case analysis reveals:

For \( S \supseteq \{ \{ B \} : B \in B \} \) it holds that

\( \Rightarrow \) \( \hat{S} \subseteq H \), or

\( \Rightarrow \) \( N_1 \) or \( N_2 \) is in \( \hat{S} \).

In case 2, one can show: CSAT(S) is NP-complete.

\( \Rightarrow \) \( H \) is the only interesting maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.

Relevance?

**Theory:** We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

**Practice:** All known applications either need only \( P \) or they need more than \( H! \)

Backtracking methods might profit from the result by reducing the branching factor.

\( \Rightarrow \) How difficult is CSAT(\( A \)) in practice?

\( \Rightarrow \) What are the relevant branching factors?
Solving general Allen CSPs

- Backtracking algorithm using **path consistency** as a forward-checking method
- Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics
  ¬¬ Which tractable fragment should one use?

Branching factors

- If the labels are split into **base relations**, then on average a label is split into **6.5 relations**
- If the labels are split into **pointizable relations** \( (P) \), then on average a label is split into **2.955 relations**
- If the labels are split into **ORD-Horn relations** \( (H) \), then on average a label is split into **2.533 relations**

¬¬ A difference of 0.422
¬¬ This makes a difference for “hard” instances.

Summary

- Allen’s interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.
- For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

4 Literature
Literature I

[Image]

J. F. Allen.
Maintaining knowledge about temporal intervals.
Also in *Readings in Knowledge Representation*.

P. van Beek and R. Cohen.
Exact and approximate reasoning about temporal relations.

B. Nebel and H.-J. Bürckert.
Reasoning about temporal relations: A maximal tractable subclass of
Allen’s interval algebra.

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Solving hard qualitative temporal reasoning problems: Evaluating the
efficiency of using the ORD-Horn class.

February 13, 2012
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Literature II

[Image]

A complete classification of complexity in Allen’s algebra in the presence
of a non-trivial basic relation.
In *Proceedings of the 17th International Joint Conference on Artificial

Reasoning about Temporal Relations: The Tractable Subalgebras of
Allen’s Interval Algebra.