Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning: Introduction

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Motivation

Quantitative vs. qualitative

Spatio-temporal configurations can be described quantitatively by specifying the coordinates of the relevant objects:

> **Example**: *At time point 10.0 object A is at position* $(11.0, 1.0, 23.7)$*, at time point 11.0 at position* $(15.2, 3.5, 23.7)$*. From time point 0.0 to 11.0, object B is at position* $(15.2, 3.5, 23.7)$*. Object C is at time point 11.0 at position* $(300.9, 25.6, 200.0)$ *and at time point 35.0 at* $(11.0, 1.0, 23.7)$*.*

Quantitative vs. qualitative

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

> **Example**: *Object A hit object B. Afterwards, object C arrived.*

Sometimes we want to reason with such descriptions, e.g.:

> *Object C was not close to object A when it hit object B.*

Representation of qualitative knowledge

Intention: Description of configurations using a finite vocabulary and reasoning about these descriptions

- Specification of a vocabulary: usually a finite set of relations (often binary) that are pairwise disjoint and exhaustive
- Specification of a language: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal semantics
- Analysis of computational properties and design of reasoning methods (often constraint propagation)
- Perhaps, specification of operational semantics for verifying whether a relation holds in a given quantitative configuration

Applications in …

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- …

Qualitative temporal relations: The Point Algebra

We want to talk about time points and binary relations over them.

- Vocabulary:
  - x equals y: $x = y$
  - x before y: $x < y$
  - x after y: $x > y$
- Language:
  - Allow for disjunctions of basic relations to express indefinite information. Use set of relations to express that. For instance, $\{<,=\}$ expresses $\leq$.
  - 2^3 different relations (including the impossible and the universal relation)
  - Use sets of atomic formulae with these relations to describe configurations. For example:

$$\{x\{=\}y, y\{<,>\}z\}$$

- Semantics: Interpret the time point symbols and relation symbols over the rational (or real) numbers.

Some reasoning problems

$$\left\{x\{<,=\}y, y\{<,=\}z, v\{<,=\}y, w\{>\}y, z\{<,=\}x\right\}$$

- Satisfiability: Are there values for all time points such that all formulae are satisfied?
- Satisfiability with $v\{=\}w$?
- Finding a satisfying instantiation of all time points
- Deduction: Does $x\{=\}y$ logically follow?
  Does $v\{<,=\}w$ follow?
- Finding a minimal description: What are the most constrained relations that describe the same set of instantiations?

Constraint Satisfaction Problems

From a logical point of view …

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory / model that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in satisfiability, model finding, and deduction

⇝ Constraint satisfaction problems

CSP: Definition

Definition

A constraint satisfaction problem (CSP) is given by

- a set V of n variables $\{v_1, \dots, v_n\}$,
- for each v_i, a value domain D_i
- constraints (relations over subsets of the variables)

Tasks:

Find one (or all) solution(s), i. e., tuples

$$(d_1, \dots, d_n) \in D_1 \times \cdots \times D_n$$

such that the assignment $v_i \mapsto d_i$ $(1 \leq i \leq n)$ satisfies all constraints.

CSP: An example

k-colorability: Can we color the nodes of a graph with k colors in a way such that all nodes connected by an edge have different colors?

- The node set is the set of variables
- The domain of each variable is $\{1, \ldots, k\}$
- The constraints are that nodes connected by an edge must have a different value

Note: This CSP has a particular restricted form:

- only binary constraints
- the domains are finite

Other examples: many problems (e.g. cross-word puzzle, n-queens problem, configuration, …) can be cast as a CSP (and solved this way)

CSP: Computational complexity

Theorem

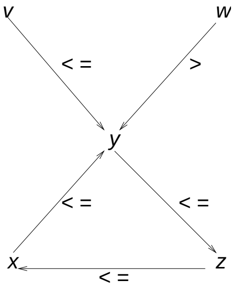
It is NP-hard to decide solvability of CSPs, even binary CSPs.

Proof.

Since k-colorability is NP-complete (even for fixed $k \geq 3$), solvability of CSPs in general must be NP-hard. □

Our example: Point relations

- Our point relation CSP is a binary CSP with infinite domains.
- It can be represented as a constraint graph:



Constraint Solving Methods

Solving CSP

- Enumeration of all assignments and testing
  $\rightsquigarrow$ … too costly
- Backtracking search
  $\rightsquigarrow$ 1001 different strategies, often “dead” search paths are explored extensively
- Constraint propagation: elimination of obviously impossible values followed by backtracking search
- Interleaving backtracking and constraint propagation
- Many other search methods, e.g., local search, stochastic search, etc.

How can we solve CSP with infinite domains?

General assumptions

- Only at most binary constraints (i.e., we can use constraint graph)
- Uniform domain D for all variables
- Unary constraints D_i and binary constraints R_{ij} are sets of values or sets of pairs of values, resp.
- We assume that for all nodes i,j:

$$(x,y) \in R_{ij} \Rightarrow (y,x) \in R_{ji}$$

Local consistency

A CSP is locally consistent if for particular subsets of the variables, solutions of the restricted CSP can be extended to solutions of a larger set of variables.
Enforcing local consistency: methods to transform a CSP into a tighter, but "equivalent" problem.

Definition

A binary CSP $\langle V, D, C \rangle$ is arc-consistent (or 2-consistent) if for all nodes $1 \leq i, j \leq n$,

$$x \in D_i \Rightarrow \exists y \in D_j \text{ s.t. } (x, y) \in R_{ij}$$

⇝ When a CSP is arc-consistent, each one variable assignment $\{v_i\} \rightarrow D$ that satisfies all (unary) constraints in v_i, i. e., D_i, can be extended to a two variable assignment $\{v_i, v_j\} \rightarrow D$ that satisfies all unary/binary constraints in these variables, i. e., D_i, D_j, and R_{ij}.

Enforcing arc consistency

EnforceArcConsistency $(\mathcal{C})$:
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but arc-consistent CSP $\mathcal{C}'$

repeat
  for each arc (v_i, v_j) with $R_{ij} \in C$
    $D_i := D_i \cap \{x \in D_i : \text{ex. } y \in D_j \text{ s.t. } (x,y) \in R_{ij}\}$
  endfor
until no domain is changed

- Terminates in time $O(n^3 \cdot k^3)$ if we have finite domains (where k is the maximal number of values in one of the domains).
- There exist different (more efficient) algorithms for enforcing arc consistency.

Arc consistency

Lemma

- *Enforcing arc consistency yields an arc-consistent CSP.*
- *Enforcing arc consistency is solution invariant, i. e. it does not change the set of solutions.*

Note: Arc-consistent CSPs need not be consistent, and vice versa.

$$D_1 = \{1, 2, 3\}$$
$$D_2 = \{2, 3\}$$
$$D_3 = \{2\}$$
$$R_{ij} = \text{“}\neq\text{''} \text{ for } i \neq j$$

1. $D_1 := D_1 \cap \{x : y \in D_3 \wedge (x, y) \in R_{13}\} = \{1, 3\}$
2. $D_2 := D_2 \cap \{x : y \in D_3 \wedge (x, y) \in R_{23}\} = \{3\}$
3. $D_1 := D_1 \cap \{x : y \in D_2 \wedge (x, y) \in R_{12}\} = \{1\}$
4. CSP is now arc-consistent

- Since all unary constraints are singletons, this defines a solution of the CSP.
- Because enforcing arc consistency does not change the set of solutions, this is the unique solution of the original CSP.

Definition

A binary CSP $\langle V, D, C\rangle$ is path-consistent (or 3-consistent) if for all nodes $1 \leq i,j,k \leq n$,

$$x \in D_i, y \in D_j, (x,y) \in R_{ij} \Rightarrow$$
$$\exists z \in D_k \text{ s. t. } (x,z) \in R_{ik} \text{ and } (y,z) \in R_{jk}$$

$\rightsquigarrow$ When a CSP is path-consistent, each two variable assignment $\{v_i, v_j\} \rightarrow D$ satisfying all constraints in v_i and v_j can be extended to any three variable assignment $\{v_i, v_j, v_k\} \rightarrow D$ such that all constraints in these variables are satisfied.

Enforcing path consistency

EnforcePathConsistency $(\mathcal{C})$:
Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$ of size n
Output: an equivalent, but path-consistent CSP $\mathcal{C}'$

repeat
  for all $1 \leq i, j, k \leq n$
    $R_{ij} := R_{ij} \cap$
      $\{(x,y) : \text{ex. } z \in D_k \text{ s.t. } (x,z) \in R_{ik} \text{ and } (y,z) \in R_{jk}\}$
  endfor
until no binary constraint is changed

- Terminates in time $O(n^5 \cdot k^5)$ if we have finite domains (where k is the maximal number of values).
- Enforcing path consistency is solution invariant.

Local consistency: k-consistency and strong k-consistency

Definition

- A binary CSP $\langle V, D, C \rangle$ is k-consistent if, given variables $x_1, \ldots, x_k$ and an assignment $a : \{x_1, \ldots, x_{k-1}\} \to D$ that satisfies all constraint in these variables, a can be extended to an assignment $a' : \{x_1, \ldots, x_k\} \to D$ that satisfies all constraints in these k variables.
- A binary CSP $\langle V, D, C \rangle$ is strongly k-consistent if it is k'-consistent for each $k' \leq k$.
- A binary CSP $\langle V, D, C \rangle$ is globally consistent if it is strongly n-consistent where n is the size of V.

Local consistency

- k-consistency: the computation costs grow exponentially with k.
- If a CSP is globally consistent, then
  - a solution can be constructed in polynomial time,
  - its constraints are minimal, and
  - it has a solution if and only if there is no empty constraint.
- k-consistent $\not\Rightarrow$ $k-1$-consistent

Qualitative Constraint Satisfaction Problems

Qualitative reasoning with CSP

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If we want to use CSPs for qualitative reasoning, we have

- infinite domains
- mostly only finitely many relations (basic relations and their unions)
- arc-consistent CSPs (usually)

Questions:

- How do we achieve k-consistency (for some fixed k)?
- Is k-consistency (for some fixed k) enough to guarantee global consistency?
- Is a CSP with only base relations always satisfiable?

Operations on binary relations

Composition:

$$R_1 \circ R_2 = \{(x,y) \in D^2 : \exists z \in D \text{ s. t. } (x,z) \in R_1 \text{ and } (z,y) \in R_2\}$$

Converse:

$$R^{-1} = \{(x,y) \in D^2 : (y,x) \in R\}$$

Intersection:

$$R_1 \cap R_2 = \{(x,y) \in D^2 : (x,y) \in R_1 \text{ and } (x,y) \in R_2\}$$

Union:

$$R_1 \cup R_2 = \{(x,y) \in D^2 : (x,y) \in R_1 \text{ or } (x,y) \in R_2\}$$

Complement:

$$\overline{R} = \{(x,y) \in D^2 : (x,y) \notin R\}$$

Conditions on vocabulary for qualitative reasoning

Let $\mathcal{B}$ be a finite set of (binary) base relations. We require:

- the relations in $\mathcal{B}$ are JEPD, i. e., jointly exhaustive and pairwise disjoint.
- $\mathcal{B}$ is closed under converse.

Let $\mathcal{A}$ be the set of relations that can be built by taking the unions of relations from $\mathcal{B}$ ($\rightsquigarrow 2^{|\mathcal{B}|}$ different relations). Then $\mathcal{A}$ is closed under converse, complement, intersection and union.

- $\mathcal{A}$ should be closed under composition of base relations, i. e., for all $B, B' \in \mathcal{B}$, $B \circ B' \in \mathcal{A}$.
  If so, $\mathcal{A}$ is closed under composition of arbitrary relations.

Note: This condition does not hold necessarily.
For example, $\mathcal{B} = \{<, =, >\}$ interpreted over the integers is not closed under composition (and has no finite closure) (see exercises).

Computing operations on relations

Let $\mathcal{A}$ be a relation system over the set of base relations $\mathcal{B}$ that satisfies the conditions spelled out above.

$\rightsquigarrow$ We may write relations as sets of base relations:

$$B_1 \cup \cdots \cup B_n \sim \{B_1, \ldots, B_n\}$$

Then the operations on the relations can be computed as follows:

Composition:

$$\{B_1, \ldots B_n\} \circ \{B'_1, \ldots, B'_m\} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (B_i \circ B'_j)$$

Converse:

$$\{B_1, \ldots, B_n\}^{-1} = \{B_1^{-1}, \ldots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \ldots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and union are defined set-theoretically.

Motivation

CSP

Solving CSP

Qualitative CSP

A Pathological Relation System

Outlook

Literature

Reasoning problems

Given a qualitative CSP:

CSP-satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-entailment (CENT):

- Given in addition xRy: Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y the strongest constrained (minimal) relation entailed by the CSP.

These problems are equivalent under Turing reductions:

Reductions between CSP problems

Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.

CSAT $\leq_T$ CENT and CENT $\leq_T$ CMIN are obvious.

CENT $\leq_T$ CSAT: We solve CENT ($CSP \models xRy?$) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations not in R. xRy is entailed by the CSP iff for all these base relations we get a negative answer.

CMIN $\leq_T$ CENT: We use entailment for computing the minimal constraint for each pair. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. $\square$

Path consistency for qualitative CSPs

Given a qualitative CSP with $R_{ij} = R_{ji}^{-1}$. Then path consistency can be enforced by doing the following:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}).$$

Path consistency guarantees ...

- sometimes minimality
- sometimes satisfiability
- however sometimes the CSP is not satisfiable, even if the CSP contains only base relations

All this depends on the vocabulary (and its interpretation).

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Example: Point Algebra

	$<$	$=$	$>$
$<$	$<$	$<$	$<,=,>$
$=$	$<$	$=$	$>$
$>$	$<,=,>$	$>$	$>$

Fig.: Composition table for the point algebra.
For example: $\{<\} \circ \{=\} = \{<\}$

- $\{<,=\} \circ \{<\} = \{<\}$
- $\{<,>\} \circ \{<\} = \{<,=,>\}$
- $\{<,=\}^{-1} = \{>,=\}$
- $\{<,=\} \cap \{>,=\} = \{=\}$

Some properties of the point relations

Theorem

A path-consistent CSP over the point relations is consistent.

Corollary

CSAT, CENT and CMIN are polynomial problems for the point relations.

Theorem

A path-consistent CSP over all point relations without $\{<,>\}$ is minimal.

Proofs later ...

A Pathological Relation System

A pathological relation system

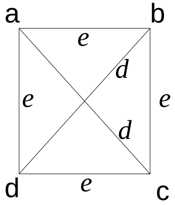
Let e, d, i be (self-converse) base relations between points on a circle:

- e: Rotation by 72 degrees (left or right)
- d: Rotation by 144 degrees (left or right)
- i: Identity

Composition table:

$$e \circ e = \{i, d\}$$
$$d \circ d = \{i, e\}$$
$$e \circ d = \{e, d\}$$
$$d \circ e = \{e, d\}$$

The following CSP is path-consistent and contains only base relations, but it is not satisfiable:



Outlook

Outlook

- Qualitative representation and reasoning usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simple logical theories consisting of sets of atomic formulae (and some background theory).
- Reasoning problems are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with CSP methods (but note: infinite domains).
- Path consistency is the basic reasoning step . . . sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.
- Can be taken further $\rightsquigarrow$ relation algebra

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