Principles of Knowledge Representation and Reasoning

Semantic Networks and Description Logics IV: Description Logics – Algorithms

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Motivation

Reasoning problems & algorithms

Reasoning problems:

- Satisfiability or subsumption of concept descriptions
- Satisfiability or instance relation in ABoxes

Solving techniques presented in this chapter:

- Structural subsumption algorithms
  - Normalization of concept descriptions and structural comparison
  - very fast, but can only be used for small DLs
- Tableau algorithms
  - Similar to modal tableau methods
  - Often the methhod of choice

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Structural Subsumption Algorithms

Structural subsumption algorithms

In what follows we consider the rather small logic $\mathcal{FL}^-$:

- $C \sqcap D$
- $\forall r.C$
- $\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

$$\forall r.C \sqcap \forall r.D \rightarrow \forall r.(C \sqcap D).$$

2. Compare all conjuncts with each other.
For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.

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Example

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$$D = \texttt{Human} \sqcap \exists\texttt{has-child} \sqcap \forall\texttt{has-child}.\texttt{Human} \sqcap \forall\texttt{has-child}.\exists\texttt{has-child}$$

$$C = \texttt{Human} \sqcap \texttt{Female} \sqcap \exists\texttt{has-child} \sqcap \forall\texttt{has-child}.(\texttt{Human} \sqcap \texttt{Female} \sqcap \exists\texttt{has-child})$$

Check: $C \sqsubseteq D$

1. Collect value restrictions in D:
   $\ldots\forall\texttt{has-child}.(\texttt{Human} \sqcap \exists\texttt{has-child})$
2. Compare:
   1. For $\texttt{Human}$ in D, we have $\texttt{Human}$ in C.
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$\rightsquigarrow$ C is subsumed by D !

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SUB(C, D) algorithm:

1. Reorder terms (using commutativity, associativity and value restriction law):

$$C = \bigsqcap A_i \sqcap \bigsqcap \exists r_j \sqcap \bigsqcap \forall r_k : C_k$$

$$D = \bigsqcap B_l \sqcap \bigsqcap \exists s_m \sqcap \bigsqcap \forall s_n : D_n$$

2. For each B_l in D, is there an A_i in C with $A_i = B_l$?
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$\leadsto$ $C \sqsubseteq D$ iff all questions are answered positively.

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Soundness

Theorem (Soundness)

$SUB(C,D) \Rightarrow C \sqsubseteq D$

Proof sketch.

Reordering of terms step (1):

1. Commutativity and associativity are trivial
2. Value restriction law. We show: $(\forall r.(C \sqcap D))^{\mathcal{I}} = (\forall r.C \sqcap \forall r.D)^{\mathcal{I}}$
   Assume $d \in (\forall r.(C \sqcap D))^{\mathcal{I}}$.
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Steps (2+3+4): Induction on the nesting depth of $\forall$-expressions. $\square$

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Theorem (Completeness)

$C \sqsubseteq D \Rightarrow SUB(C, D)$.

Proof idea.

One shows the contrapositive:

$$\neg \mathrm{SUB}(C, D) \Rightarrow C \not\sqsubseteq D$$

Idea: If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element d such that

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- $\neg A$ (atomic negation),
- $(\leq n r)$, $(\geq n r)$ (cardinality restrictions),
- $r \circ s$ (role composition)

do not lead to any problems.

However: If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.

More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

Reason: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).

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Reason: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).

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Extensions of $\mathcal{FL}^-$ by

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ABox reasoning

Idea: Abstraction + classification

- Complete ABox by propagating value restrictions to role fillers.
- Compute for each object its most specialized concepts.
- These can then be handled using the ordinary subsumption algorithm.

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Tableau Subsumption Method

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Logic $\mathcal{ALC}$:

- $C \sqcap D$
- $C \sqcup D$
- $\neg C$
- $\forall r.C$
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Idea: Decide (un-)satisfiability of a concept description C by trying to systematically construct a model for C.
If that is successful, C is satisfiable. Otherwise, C is unsatisfiable.

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Example

TBox:

$$\texttt{Hermaphrodite} \doteq \texttt{Male} \sqcap \texttt{Female}$$

$$\texttt{Parents-of-sons-and-daughters} \doteq \exists\texttt{has-child.Male} \sqcap \exists\texttt{has-child.Female}$$

$$\texttt{Parents-of-hermaphrodite} \doteq \exists\texttt{has-child.Hermaphrodite}$$

Query:

$$\texttt{Parents-of-sons-and-daughters} \sqsubseteq_{\mathcal{T}} \texttt{Parents-of-hermaphrodites}$$

Example: Subsumption in a TBox

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Reductions

1. Unfolding:
   $$\exists \texttt{has-child}.\texttt{Male} \sqcap \exists \texttt{has-child}.\texttt{Female} \sqsubseteq \exists \texttt{has-child}.(\texttt{Male} \sqcap \texttt{Female})$$
2. Reduction to unsatisfiability: Is the concept
   $$\exists \texttt{has-child}.\texttt{Male} \sqcap \exists \texttt{has-child}.\texttt{Female} \sqcap \neg \exists \texttt{has-child}(\texttt{Male} \sqcap \texttt{Female})$$
   unsatisfiable?
3. Negation normal form (move negations inside):
   $$\exists \texttt{has-child}.\texttt{Male} \sqcap \exists \texttt{has-child}.\texttt{Female} \sqcap \forall \texttt{has-child}.(\neg \texttt{Male} \sqcup \neg \texttt{Female})$$
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UNI FREIBURG

Model construction (1)

1. Assumption: There exists an object x in the interpretation of our concept:
$$x \in (\exists \ldots)^{\mathcal{I}}$$

2. This implies that x is in the interpretation of all conjuncts:
$$x \in (\exists \texttt{has-child.Male})^{\mathcal{I}}$$
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3. This implies that there should be objects y and z such that $(x, y) \in \texttt{has-child}^{\mathcal{I}}$, $(x, z) \in \texttt{has-child}^{\mathcal{I}}$, $y \in \texttt{Male}^{\mathcal{I}}$ and $z \in \texttt{Female}^{\mathcal{I}}$, and ...

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$x : \exists \texttt{has-child.Male}$

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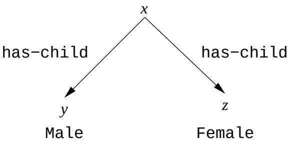

Model construction (3)

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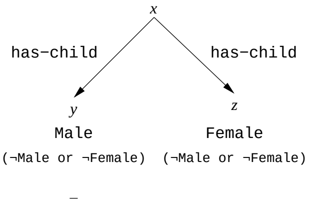

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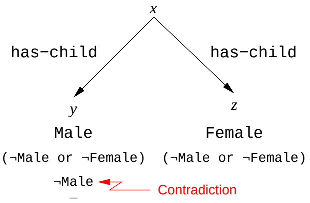

Model construction (5)

$x : \exists$has-child.Male

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$x : \forall$has-child.$(\neg$Male $\sqcup \neg$Female$)$

$y : \neg$Female

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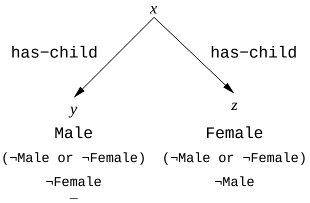

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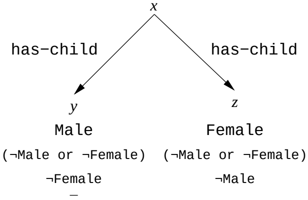

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Tableau method (1): NNF

We write: $C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$. Now we have the following equivalences:

$$\neg(C \sqcap D) \equiv \neg C \sqcup \neg D \qquad \neg(C \sqcup D) \equiv \neg C \sqcap \neg D$$
$$\neg(\forall r.C) \equiv \exists r.\neg C \qquad \neg(\exists r.C) \equiv \forall r.\neg C$$
$$\neg\neg C \equiv C$$

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).

Theorem (NNF)

The negation normal form of an $\mathcal{ALC}$ concept can be computed in polynomial time.

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Tableau method (2): Constraint systems

A constraint is a syntactical object of the form:

$$x: C \quad \text{or} \quad x\,r\,y,$$

where C is a concept description in NNF, r is a role name, and x and y are variable names.

Let $\mathcal{I}$ be an interpretation with universe $\mathcal{D}$. An $\mathcal{I}$-assignment α is a function that maps each variable symbol to an object of the universe $\mathcal{D}$.

A constraint $x: C$ ($x\,r\,y$) is satisfied by an $\mathcal{I}$-assignment α if $\alpha(x) \in C^{\mathcal{I}}$ (resp. $(\alpha(x), \alpha(y)) \in r^{\mathcal{I}}$).

Tableau method (2): Constraint systems

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Tableau method (3): Constraint systems

Definition

A constraint system S is a finite, non-empty set of constraints.
An $\mathcal{I}$-assignment α satisfies S if α satisfies each constraint in S.
S is satisfiable if there exist $\mathcal{I}$ and α such that α satisfies S.

Theorem

An $\mathcal{ALC}$ concept C in NNF is satisfiable if and only if the system $\{x : C\}$ is satisfiable.

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Tableau method (4): Transforming constraint systems

Transformation rules:

1. $S \rightarrow_{\sqcap} \{x: C_1, x: C_2\} \cup S$
   if $(x: C_1 \sqcap C_2) \in S$ and either $(x: C_1)$ or $(x: C_2)$ or both are not in S.
2. $S \rightarrow_{\sqcup} \{x: D\} \cup S$
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Notice: Deterministic rules (1,3,4) vs. non-deterministic (2).

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Generating rules (3) vs. non-generating (1,2,4).

Tableau method (5): Invariances

Theorem (Invariance)

Let S and T be constraint systems.

1. *If T has been generated by applying a deterministic rule to S, then S is satisfiable if and only of T is satisfiable.*
2. *If T has been generated by applying a non-deterministic rule to S, then S is satisfiable if T is satisfiable. Furthermore, if a non-deterministic rule can be applied to S, then it can be applied such that S is satisfiable if and only if the resulting system T is satisfiable.*

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Tableau method (6): Soundness and completeness

A constraint system is called closed if no transformation rule can be applied.

A clash is a pair of constraints of the form $x : A$ and $x : \neg A$, where A is a concept name.

Theorem (Soundness and Completeness)

A closed constraint system is satisfiable if and only it does not contain a clash.

Proof idea.

$\Rightarrow$: obvious. $\Leftarrow$: Construct a model by using the concept labels. $\square$

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Space requirements

Because the tableau method is non-deterministic ($\rightarrow_\sqcup$ rule), there could be exponentially many closed constraint systems in the end.

Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

Example

$$\begin{array}{l} \exists r.A \sqcap \exists r.B \sqcap \\ \forall r.(\ \exists r.A \sqcap \exists r.B \sqcap \\ \quad \forall r.(\ \exists r.A \sqcap \exists r.B \sqcap \\ \quad\quad \forall r.(\ldots))) \end{array}$$

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ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $x \neq y$ (for UNA):

- Normalize and unfold and add inequalities for all pairs of objects mentioned in the ABox.
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