Description Logics – Algorithms

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1 Motivation
Reasoning problems & algorithms

Reasoning problems:

- **Satisfiability** or **subsumption** of concept descriptions
- **Satisfiability** or **instance relation** in ABoxes

Solving techniques presented in this chapter:

- **Structural subsumption algorithms**
  - **Normalization** of concept descriptions and **structural comparison**
  - very fast, but can only be used for small DLs
- **Tableau algorithms**
  - Similar to modal tableau methods
  - Often the method of choice
2 Structural Subsumption Algorithms

- Idea
- Example
- Algorithm
- Soundness
- Completeness
- Generalizations
- ABox Reasoning
Structural subsumption algorithms

In what follows we consider the rather small logic $\mathcal{FL}^-$:

- $C \sqcap D$
- $\forall r.C$
- $\exists r$ (simple existential quantification)

To solve the subsumption problem for this logic we apply the following idea:

1. In the conjunction, collect all universally quantified expressions (also called value restrictions) with the same role and build complex value restriction:

   $$\forall r.C \sqcap \forall r.D \rightarrow \forall r.(C \sqcap D).$$

2. Compare all conjuncts with each other.
   For each conjunct in the subsuming concept there should be a corresponding one in the subsumed one.
Example

$D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child. Human} \sqcap \forall \text{has-child.} \exists \text{has-child}$

$C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child.}(\text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child})$

Check: $C \subseteq D$

1. **Collect** value restrictions in $D$:
   $\ldots \forall \text{has-child.}(\text{Human} \sqcap \exists \text{has-child})$

2. **Compare**:
   1. For Human in $D$, we have Human in $C$.
   2. For $\exists \text{has-child}$ in $D$, we have $\exists \text{has-child}$ in $C$.
   3. For $\forall \text{has-child.}(\ldots)$ in $D$, we have Human and $\exists \text{has-child}$ in $C$.

$\implies C$ is subsumed by $D$!
Subsumption algorithm

**SUB(C, D) algorithm:**

1. Reorder terms (using **commutativity**, **associativity** and **value restriction law**):

   \[ C = \bigcap A_i \cap \bigcap \exists r_j \cap \bigcap \forall r_k : C_k \]

   \[ D = \bigcap B_l \cap \bigcap \exists s_m \cap \bigcap \forall s_n : D_n \]

2. For each \( B_l \) in \( D \), is there an \( A_i \) in \( C \) with \( A_i = B_l \)?

3. For each \( \exists s_m \) in \( D \), is there an \( \exists r_j \) in \( C \) with \( s_m = r_j \)?

4. For each \( \forall s_n : D_n \) in \( D \), is there a \( \forall r_k : C_k \) in \( C \) such that \( s_n = r_k \) and \( C_k \sqsubseteq D_n \) (i.e., check \( \text{SUB}(C_k, D_n) \))? 

\[ \iff C \sqsubseteq D \text{ iff all questions are answered positively.} \]
Soundness

Theorem (Soundness)

\[ \text{SUB}(C, D) \Rightarrow C \subseteq D \]

Proof sketch.

Reordering of terms step (1):

1. Commutativity and associativity are trivial
2. Value restriction law. We show: \((\forall r. (C \cap D))^I = (\forall r. C \cap \forall r. D)^I\)

Assume \(d \in (\forall r. (C \cap D))^I\).

If there is no \(e \in D\) with \((d, e) \in r^I\) it follows trivially that \(d \in (\forall r. C \cap \forall r. D)^I\).

If there is an \(e \in D\) with \((d, e) \in r^I\) it follows \(e \in (C \cap D)^I = C^I \cap D^I\).

Since \(e\) is arbitrary, we have \(d \in (\forall r. C)^I\) and \(d \in (\forall r. D)^I\), i.e., \((\forall r. (C \cap D))^I \subseteq (\forall r. C \cap \forall r. D)^I\).

The other direction is similar.

Steps (2+3+4): Induction on the nesting depth of \(\forall\)-expressions.
Completeness

Theorem (Completeness)

\[ C \sqsubseteq D \Rightarrow \text{SUB}(C, D). \]

Proof idea.

One shows the contrapositive:

\[ \neg\text{SUB}(C, D) \Rightarrow C \not\sqsubseteq D \]

**Idea:** If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element \( d \) such that

\[ d \in C^\mathcal{I}, \text{ but } d \notin D^\mathcal{I}. \]
Generalizing the algorithm

Extensions of $\mathcal{FL}^-$ by

- $\neg A$ (atomic negation),
- $(\leq nr), (\geq nr)$ (cardinality restrictions),
- $r \circ s$ (role composition)

do not lead to any problems.

However: If we use full existential restrictions, then it is very unlikely that we can come up with a simple structural subsumption algorithm – having the same flavor as the one above.

More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison.

Reason: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).
Idea: Abstraction + classification

- **Complete** ABox by propagating value restrictions to role fillers.
- Compute for each object its **most specialized concepts**.
- These can then be handled using the ordinary subsumption algorithm.
3 Tableau Subsumption Method

- Example
- Reductions: Unfolding & Unsatisfiability
- Model Construction
- Equivalences & NNF
- Constraint Systems
- Transforming Constraint Systems
- Invariances
- Soundness and Completeness
- Space Complexity
- ABox Reasoning
Tableau method

Logic $\mathcal{ALC}$:

- $C \sqcap D$
- $C \sqcup D$
- $\neg C$
- $\forall r.C$
- $\exists r.C$

Idea: Decide (un-)satisfiability of a concept description $C$ by trying to systematically construct a model for $C$. If that is successful, $C$ is satisfiable. Otherwise, $C$ is unsatisfiable.
Example: Subsumption in a TBox

Example

TBox:

\[
\text{Hermaphrodite} \equiv \text{Male} \sqcap \text{Female} \\
\text{Parents-of-sons-and-daughters} \equiv \exists \text{has-child}.\text{Male} \sqcap \exists \text{has-child}.\text{Female} \\
\text{Parents-of-hermaphrodite} \equiv \exists \text{has-child}.\text{Hermaphrodite}
\]

Query:

\[
\text{Parents-of-sons-and-daughters} \sqsubseteq \text{T} \\
\text{Parents-of-hermaphrodites}
\]
Reductions

1 Unfolding:
\[ \exists \text{has-child} \cdot \text{Male} \sqcap \exists \text{has-child} \cdot \text{Female} \sqsubseteq \exists \text{has-child} \cdot (\text{Male} \sqcap \text{Female}) \]

2 Reduction to unsatisfiability: Is the concept
\[ \exists \text{has-child} \cdot \text{Male} \sqcap \exists \text{has-child} \cdot \text{Female} \sqcap \neg \exists \text{has-child} \cdot (\text{Male} \sqcap \text{Female}) \]
unsatisfiable?

3 Negation normal form (move negations inside):
\[ \exists \text{has-child} \cdot \text{Male} \sqcap \exists \text{has-child} \cdot \text{Female} \sqcap \forall \text{has-child} \cdot (\neg \text{Male} \sqcup \neg \text{Female}) \]

4 Try to construct a model
1 Assumption: There exists an object $x$ in the interpretation of our concept:

$$x \in (\exists \ldots)^\mathcal{I}$$

2 This implies that $x$ is in the interpretation of all conjuncts:

$$x \in (\exists \text{has-child}.\text{Male})^\mathcal{I}$$

$$x \in (\exists \text{has-child}.\text{Female})^\mathcal{I}$$

$$x \in (\forall \text{has-child}.(\neg\text{Male} \sqcup \neg\text{Female}))^\mathcal{I}$$

3 This implies that there should be objects $y$ and $z$ such that

$(x,y) \in \text{has-child}^\mathcal{I}$, $(x,z) \in \text{has-child}^\mathcal{I}$, $y \in \text{Male}^\mathcal{I}$ and $z \in \text{Female}^\mathcal{I}$, and ...
Model construction (2)

\[ x : \exists \text{has-child}.\text{Male} \]
\[ x : \exists \text{has-child}.\text{Female} \]
Model construction (3)

\[ x : \exists \text{has-child}. \text{Male} \]
\[ x : \exists \text{has-child}. \text{Female} \]
\[ x : \forall \text{has-child}. (\neg \text{Male} \sqcup \neg \text{Female}) \]
Model construction (4)

$x: \exists \text{has-child}. \text{Male}$

$x: \exists \text{has-child}. \text{Female}$

$x: \forall \text{has-child}.(\neg \text{Male} \sqcup \neg \text{Female})$

$y: \neg \text{Male}$

![Diagram]

- $x$
- $\text{has-child}$
- $\text{has-child}$
- $y$
- $z$
- Male
- Female
- $(\neg \text{Male} \lor \neg \text{Female})$
- $(\neg \text{Male} \lor \neg \text{Female})$
- $\neg \text{Male}$
- Contradiction
Model construction (5)

\[
\begin{align*}
x &: \exists \text{has-child.} \text{Male} \\
x &: \exists \text{has-child.} \text{Female} \\
x &: \forall \text{has-child.} (\neg \text{Male} \sqcup \neg \text{Female}) \\
y &: \neg \text{Female} \\
z &: \neg \text{Male}
\end{align*}
\]

\[\Rightarrow\]

Model constructed!
Tableau method (1): NNF

We write: $C \equiv D$ iff $C \subseteq D$ and $D \subseteq C$. Now we have the following equivalences:

\[
\neg (C \land D) \equiv \neg C \lor \neg D \quad \neg (C \lor D) \equiv \neg C \land \neg D \\
\neg (\forall r. C) \equiv \exists r. \neg C \quad \neg (\exists r. C) \equiv \forall r. \neg C \\
\neg \neg C \equiv C
\]

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: negation normal form (NNF).

**Theorem (NNF)**

The negation normal form of an $\mathcal{ALC}$ concept can be computed in polynomial time.
A constraint is a syntactical object of the form:

\[ x : C \quad \text{or} \quad x r y, \]

where \( C \) is a concept description in NNF, \( r \) is a role name, and \( x \) and \( y \) are variable names.

Let \( \mathcal{I} \) be an interpretation with universe \( \mathcal{D} \). An \( \mathcal{I} \)-assignment \( \alpha \) is a function that maps each variable symbol to an object of the universe \( \mathcal{D} \).

A constraint \( x : C (x r y) \) is satisfied by an \( \mathcal{I} \)-assignment \( \alpha \) if \( \alpha(x) \in C^\mathcal{I} \) (resp. \( (\alpha(x), \alpha(y)) \in r^\mathcal{I} \)).
Tableau method (3): Constraint systems

Definition

A constraint system $S$ is a finite, non-empty set of constraints. An $\mathcal{I}$-assignment $\alpha$ satisfies $S$ if $\alpha$ satisfies each constraint in $S$. $S$ is satisfiable if there exist $\mathcal{I}$ and $\alpha$ such that $\alpha$ satisfies $S$.

Theorem

An $\mathcal{ALC}$ concept $C$ in NNF is satisfiable if and only if the system \( \{x : C\} \) is satisfiable.
Tableau method (4): Transforming constraint systems

Transformation rules:

1. \[ S \rightarrow \cap \{x : C_1, x : C_2\} \cup S \]
   if \((x : C_1 \cap C_2) \in S\) and either \((x : C_1)\) or \((x : C_2)\) or both are not in \(S\).

2. \[ S \rightarrow \cup \{x : D\} \cup S \]
   if \((x : C_1 \cup C_2) \in S\) and neither \((x : C_1) \in S\) nor \((x : C_2) \in S\) and \(D = C_1\) or \(D = C_2\).

3. \[ S \rightarrow \exists \{xry, y : C\} \cup S \]
   if \((x : \exists r.C) \in S\), \(y\) is a fresh variable, and there is no \(z\) s.t. \((xrz) \in S\) and \((z : C) \in S\).

4. \[ S \rightarrow \forall \{y : C\} \cup S \]
   if \((x : \forall r.C), (xry) \in S\) and \((y : C) \notin S\).

Notice: Deterministic rules (1,3,4) vs. non-deterministic (2).

Generating rules (3) vs. non-generating (1,2,4).
Theorem (Invariance)

Let $S$ and $T$ be constraint systems.

1. If $T$ has been generated by applying a deterministic rule to $S$, then $S$ is satisfiable if and only if $T$ is satisfiable.

2. If $T$ has been generated by applying a non-deterministic rule to $S$, then $S$ is satisfiable if $T$ is satisfiable. Furthermore, if a non-deterministic rule can be applied to $S$, then it can be applied such that $S$ is satisfiable if and only if the resulting system $T$ is satisfiable.

Theorem (Termination)

Let $C$ be an ALC concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{ x : C \}$. 
A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form \( x : A \) and \( x : \neg A \), where \( A \) is a concept name.

**Theorem (Soundness and Completeness)**

A closed constraint system is satisfiable if and only if it does not contain a clash.

**Proof idea.**

\( \Rightarrow \): obvious. \( \Leftarrow \): Construct a model by using the concept labels.
Space requirements

Because the tableau method is non-deterministic ($\to \Box$ rule), there could be exponentially many closed constraint systems in the end.

Interestingly, applying the rules on a single constraint system can lead to constraint systems of exponential size.

Example

$$\exists r. A \sqcap \exists r. B \sqcap$$

$$\forall r. (\exists r. A \sqcap \exists r. B \sqcap$$

$$\forall r. (\exists r. A \sqcap \exists r. B \sqcap$$

$$\forall r. (\ldots))$$

However: One can modify the algorithm so that it needs only polynomial space.

Idea: Generate a $y$ only for one $\exists r. C$ and then proceed into the depth.
ABox reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form \( x \neq y \) (for UNA):

- Normalize and unfold and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in \( \mathcal{ALC} \) we do not need this because we are never forced to identify two objects.
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