Principles of Knowledge Representation and Reasoning

Dynamics of belief

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Introduction

Principles

Propositional logic flaws:

- The world is not always static.
- The knowledge about the world is sometimes uncertain or imprecise

Therefore:

- Need the possibility to incorporate new (possibly contradictory) beliefs;
- Need to take into account change in the world;

The Guettier argument

Plato - Theaetetus: A knowledge (a rightful opinion) is a piece of

1. Justified True Belief

Agrippa's trilemma - A problem with the justification:

1. Either the justification stops to some unjustified belief;
2. The justification is infinite (Socrates' clouds);
3. The justification is supported by affirmations it is supposed to justify (Baron Münchhausen's hair).

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Foundationalism and coherentism

Three solutions:

Foundationalism Allow for unjustified beliefs

$\rightarrow$ Formalization issues

$\rightarrow$ Humans don't keep track of sources

$\rightarrow$ TMS System

"Infinitism" Allow for infinite justification

$\rightarrow$ Does it really make sense?

Coherentism Allow for circular justifications

$\rightarrow$ What is a solid belief?

$\rightarrow$ Belief revision/update

- In any cases, information is extremely important and should not be discarded carelessly.

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Social choice theory: the Arrow theorem

Arrow's impossibility theorem - there is no voting system which respects:

- Non-dictatorship
  (all voters should be taken into account);
- Universality
  (complete and deterministic ranking);
- Independance of irrelevant alternatives
  (ranking between x and y depends only on x and y);
- Pareto efficiency
  (if all preferences states $x < y$, then so must the results).

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Revision or update

- We have a theory about the world, and the new information is meant to correct our theory

⇝ belief revision: change your belief state minimally in order to accommodate the new information

- We have a (supposedly) correct theory about the current state of the world, and the new information is meant to record a change in the world

⇝ belief update: incorporate the change by assuming that the world has changed minimally

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Update and revision are different

Assume the new information is consistent with our old beliefs.

- In case of belief revision, we would like to add the new information monotonically to our old beliefs.
- For belief update this is not necessarily the case.
  - Assume we know that the **door is open or the window is open**.
  - Assume we learn that the world has changed and the **door is now closed**.
  - In this case, we do not want to add this information monotonically to our theory, since we would be forced to conclude that **the window is open**.

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Overview of an operation

What are the criteria for definition of a belief revision operation?

Gärdenfors and Rott - belief revision (1995):

1. How are beliefs represented?
2. What is the relation between beliefs represented explicitly in the belief base and beliefs which can be derived from them?
3. In the face of a contradiction, how to deal with both new and old information?

Belief base, belief set or interpretation?

General assumption:

- A belief set is a deductively closed theory, i.e., $K = \mathrm{Cn}(K)$ with Cn the consequence operator
- $\mathcal{L}$: logical language (propositional logic)
- $\mathrm{Th}_{\mathcal{L}}$: set of deductively closed theories (or belief sets) over $\mathcal{L}$

Belief change operations

Monotonic addition: $+ : \mathrm{Th}_{\mathcal{L}} \times \mathcal{L} \rightarrow \mathrm{Th}_{\mathcal{L}}$

$K + \psi = \mathrm{Cn}(K \cup \{\psi\})$

Revision: $\dot{+} : \mathrm{Th}_{\mathcal{L}} \times \mathcal{L} \rightarrow \mathrm{Th}_{\mathcal{L}}$

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Semantic or syntactic

Consider $K = \{a, b\}$ and $K' = \{a \wedge b\}$. What is happening to $K \dot{+} \{\neg a\}$?

Semantic

- No difference between K and K'

a	b	$\mathcal{I}$
0	0	0
0	1	0
1	0	0
1	1	1

Syntactic

- $X = \{b\}$ is the only maximal subset of K s.t. $X \cup \{\neg a\}$ is consistent.
- $X' = \emptyset$ is the only maximal subset of K' s.t. $X' \cup \{\neg a\}$ is consistent.

Belief revision

What is a good revision operator?

- Consistency: a revision has to produce a consistent set of beliefs;
- Minimality of change: a revision has to change the fewest possible beliefs;
- Priority to the new information: the 'new' information is considered more important than the 'old' one.

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AGM postulates:

($\dot{+}$1) $K \dot{+} \varphi \in \mathrm{Th}_{\mathcal{L}}$;

($\dot{+}$2) $\varphi \in K \dot{+} \varphi$;

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Characterization for belief sets' revision

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The Levi identity

Revision can be defined in terms of two suboperations.

- $+$ (expansion) denotes the simple union of beliefs;
- $-$ (contraction) denotes the removal of information contradicting the input.

The Levi identity

$$K \dot{+} \varphi \equiv Cn[(K - \neg\varphi) + \varphi]$$

Example

$$K = \{a, a \rightarrow b\} \qquad \varphi\{\neg b\}?$$

$$K - \neg\varphi = \{a\} \text{ or } \{a \rightarrow b\}$$

$$K \dot{+} \neg\varphi = \{a, \neg b\} \text{ or } \{a \rightarrow b, \neg b\}$$

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Definition

We denote by $K \perp \varphi$ the set of maximal (wrt set-theoretic inclusion) subsets J of K such that $J \not\vdash \varphi$.

Definition

Full-meet contraction is defined by $K - \varphi = \bigcap(K \perp \varphi)$.

Is full-meet contraction reasonable?

- No! It is far too cautious.
- It can nevertheless be used as a lower bound to any reasonable operator.

$K \dotplus \varphi = \bigcap(K \perp \varphi) + \varphi$ is referred to as the full-meet revision.

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Proposition

Full-meet revision respects all AGM postulates.

Proof

($\dot{+}1$) and ($\dot{+}2$) are true by construction

($\dot{+}3$) Two cases: (1) If $K + \varphi$ is consistent then $K - \varphi = K$ and $K \dot{+} \varphi = K + \varphi$. (2) If $K + \varphi$ is inconsistent then $K + \varphi = \mathrm{Cn}(\bot)$ and $K \dot{+} \varphi \subseteq K + \varphi$.

($\dot{+}4$) Because $K \not\vdash \neg\varphi$ then $K \bot \varphi = \{K\}$ and thus $K \dot{+} \varphi = K + \varphi$.

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($\dot{+}6$) Lets assume that $\alpha \in K \bot \varphi$ but $\alpha \notin K \bot \Psi$. Two cases: (1) $\alpha \cup \Psi \vdash \bot \xrightarrow{(\varphi \leftrightarrow \Psi)} \alpha \cup \varphi \vdash \bot$ which is not possible. (2) $\exists \beta$ s.t. $\alpha \subsetneq \beta$ and $\beta \cup \Psi \not\vdash \bot \xrightarrow{(\varphi \leftrightarrow \Psi)} \beta \cup \varphi \not\vdash \bot$ which is not possible.

($\dot{+}7$) and ($\dot{+}8$) Left as exercises...

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($\dot{+}$3) Two cases: (1) If $K+\varphi$ is consistent then $K-\varphi = K$ and $K\dot{+}\varphi = K+\varphi$. (2) If $K+\varphi$ is inconsistent then $K+\varphi = \mathrm{Cn}(\bot)$ and $K\dot{+}\varphi \subseteq K+\varphi$.

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($\dot{+}$5) $K\dot{+}\varphi = \mathrm{Cn}\big(\cap_{\alpha\in(K\bot\varphi)}\alpha\cup\varphi\big)$. But $\forall\alpha, \alpha\cup\varphi \not\vdash \bot$, therefore $\cap_{\alpha\in(K\bot\varphi)}\alpha\cup\varphi \not\vdash \bot$ (as PL is monotonic).

($\dot{+}$6) Lets assume that $\alpha \in K\bot\varphi$ but $\alpha \notin K\bot\Psi$. Two cases: (1) $\alpha\cup\Psi \vdash \bot \overset{(\varphi\leftrightarrow\Psi)}{\longrightarrow} \alpha\cup\varphi\vdash\bot$ which is not possible. (2) $\exists\beta$ s.t. $\alpha \subsetneq \beta$ and $\beta\cup\Psi\not\vdash\bot \overset{(\varphi\leftrightarrow\Psi)}{\longrightarrow} \beta\cup\varphi\not\vdash\bot$ which is not possible.

($\dot{+}$7) and ($\dot{+}$8) Left as exercises...

Proposition

Full-meet revision respects all AGM postulates.

Proof

($\dot{+}1$) and ($\dot{+}2$) are true by construction

($\dot{+}3$) Two cases: (1) If $K+\varphi$ is consistent then $K-\varphi=K$ and $K\dot{+}\varphi=K+\varphi$. (2) If $K+\varphi$ is inconsistent then $K+\varphi=\mathrm{Cn}(\bot)$ and $K\dot{+}\varphi\subseteq K+\varphi$.

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Proposition

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Proof

($\dotplus$1) and ($\dotplus$2) are true by construction

($\dotplus$3) Two cases: (1) If $K+\varphi$ is consistent then $K-\varphi=K$ and $K\dotplus\varphi=K+\varphi$. (2) If $K+\varphi$ is inconsistent then $K+\varphi=\mathrm{Cn}(\bot)$ and $K\dotplus\varphi\subseteq K+\varphi$.

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($\dotplus$7) and ($\dotplus$8) Left as exercises...

Maxi-choice contraction

On the other side, one can ask for the principle of minimality to be strictly respected.

Definition

A selection function for K is a function γ such that for all sentences φ:

1. If $K \perp \varphi$ is non-empty, then $\gamma(K \perp \varphi)$ is a non-empty subset of $K \perp \varphi$, and
2. If $K \perp \varphi$ is empty, then $\gamma(K \perp \varphi) = \{K\}$.

Definition

Maxichoice contraction is defined as $K - \varphi = \gamma(K \perp \varphi)$ where γ is a selection function.

Partial-meet contraction

Maxi-choice can be too bold: there is sometimes no reason to trust one piece more than one another.

Definition

A partial-meet revision operation is an operation defined as:

$$K \dot{+} \varphi = \bigcap \gamma(K \perp \varphi) + \varphi$$

Seems to be a good compromise between full-meet and maxi-choice

Distance-based revision operations

Definition

The Dalal revision operation, denoted by $\dot{+}_D$, is defined as:

$$K \dot{+}_D \varphi = \min(extMod(\varphi), \leq_K)$$

where d_H is the Hamming Distance and

$\alpha \leq_K \beta$ iff $\exists \omega \in extMod(K), \forall \omega' \in extMod(K), d_H(\alpha, \omega) \leq d_H(\beta, \omega')$

Example

	a	b	c
$\mathcal{I}_{\varphi_1}$	0	0	0
$\mathcal{I}_{\varphi_2}$	0	0	1
	0	1	0
$\mathcal{I}_{K_1}$	0	1	1
	1	0	0
$\mathcal{I}_{K_2}$	1	0	1
	1	1	0
$\mathcal{I}_{K_3}$	1	1	1

Let $\varphi = \{\neg a, \neg b\}$ and $K = \{(a \vee b) \wedge c\}$:

$$d(\mathcal{I}_{\varphi_1}, \mathcal{I}_{K_1}) = 2 \quad d(\mathcal{I}_{\varphi_2}, \mathcal{I}_{K_1}) = 1$$
$$d(\mathcal{I}_{\varphi_1}, \mathcal{I}_{K_2}) = 2 \quad d(\mathcal{I}_{\varphi_2}, \mathcal{I}_{K_2}) = 1$$
$$d(\mathcal{I}_{\varphi_1}, \mathcal{I}_{K_3}) = 3 \quad d(\mathcal{I}_{\varphi_2}, \mathcal{I}_{K_3}) = 2$$

Some complexity result

Formula-based approaches

The question does Ψ belongs to $K \dotplus \varphi$ (if $\dotplus$ is a full-meet revision operator) is $\Delta_2^p - (\Sigma_1^p \cup \Pi_1^p)$ provided that NP $\neq$ co-NP.

proof

If $\dotplus$ is a full-meet revision, $\Psi \in \mathrm{Cn}(K) \dotplus \varphi$ can be solved by the following algorithm: if $K \not\models \neg\Psi$, then $K \cup \Psi \models \varphi$ else $\Psi \models \varphi \longrightarrow$ Membership in Δ_2^p.

Furthermore, SAT can be polynomially transformed to full-meet revision by solving $\Psi \in \mathrm{Cn}(\Psi) \dotplus \top$ and UNSAT can be polynomially transform to full-meet revision by solving $\bot \in \mathrm{Cn}(\emptyset) \dotplus \Psi$. Hence, assuming that full-meet revision belongs to both NP and co-NP would lead to NP = co-NP.

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Several sources - belief merging

Principles of belief merging

There is not only one source for the information:

- Voting procedure;
- Expert system;
- Distributed databases;
- multisource knowledge acquisition.

Constructing a belief base which represents the several sources and which:

- solves the contradiction;
- reduces the redundancies;
- is consistent.

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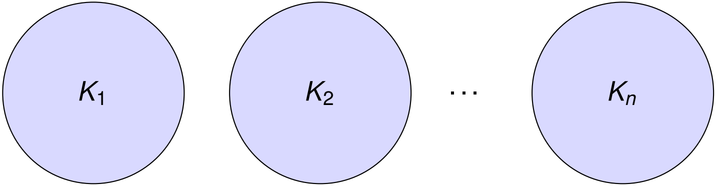
Merging in the general case



$E = \{K_1, K_2, \ldots, K_n\}$

Each K_i is consistent

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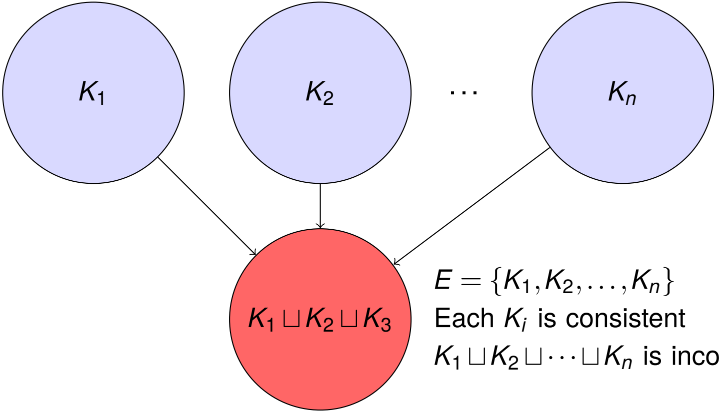
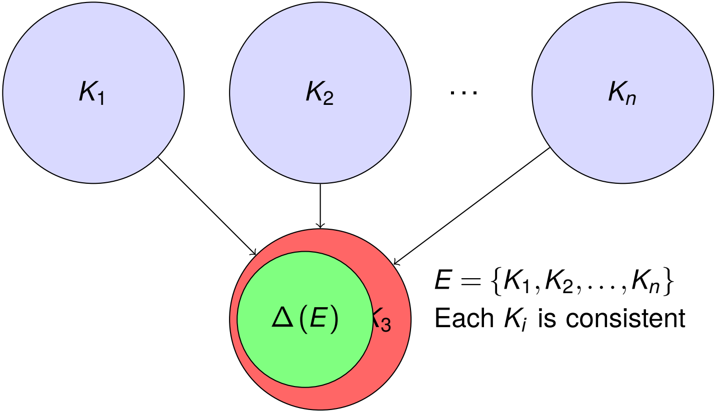


$E = \{K_1, K_2, \ldots, K_n\}$

Each K_i is consistent

$K_1 \sqcup K_2 \sqcup \cdots \sqcup K_n$ is inconsistent

Merging in the general case



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Formal framework

General assumption:

- $K_1, \ldots, K_n$ are belief bases;
- $E = \{K_1, \ldots, K_n\}$ is a multi-set of belief bases and is called a belief profile;
- IC is a propositional formula standing for constraints;
- $\sqcup$ stands for multi-set union.

Operation

Belief merging operation: $\Delta \; : \; \mathcal{L}^n \times \mathcal{L} \rightarrow \mathcal{L}$
Sometimes also called fusion operation.

Konieczny-PinoPerez postulates

(KP0) $\Delta_{IC}(E) \models IC$.

(KP1) If IC is consistent, then $\Delta_{IC}(E)$ is consistent.

(KP2) If $\bigwedge E \wedge IC$ is consistent, then $\Delta_{IC}(E) = \bigwedge E \wedge IC$.

(KP3) If $E_1 \equiv E_2$ and $IC_1 \equiv IC_2$, then $\Delta_{IC_1}(E_1) \equiv \Delta_{IC_2}(E_2)$.

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Konieczny-PinoPerez postulates

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Arbitration or majority operations

Arbitration (Arb)

$$\left.\begin{array}{l} \Delta_{IC_1}(K_1) \leftrightarrow \Delta_{IC_2}(K_2) \\ \Delta_{IC_1 \leftrightarrow \neg IC_2}(K_1 \sqcup K_2) \leftrightarrow (IC_1 \leftrightarrow \neg IC_2) \\ IC_1 \neg \vdash IC_2 \\ IC_2 \neg \vdash IC_1 \end{array}\right\} \Rightarrow \Delta_{IC_1 \vee IC_2}(K_1 \sqcup K_2) \leftrightarrow \Delta_{IC_1}(K_1)$$

Majority (Maj)

$$\exists n, \Delta_{IC}(K_1 \sqcup K_2^n) \vdash \Delta_{IC}(K_2)$$

Independence from majority (IM)

$$\forall n, \Delta_{IC}(K_1 \sqcup K_2^n) \leftrightarrow \Delta_{IC}(K_1 \sqcup K_2)$$

Link between (IM) the KP postulates

Theorem

There exists no merging operator satisfying all the KP postulates and (IM).

Proof

Consider $E_1 = \{K, \neg K\}$ and $E_2 = \{K\}$ be two belief profiles.
(IM) leads to $\Delta_\top(E_1 \sqcup E_2) = \Delta_\top(E_1)$.
(KP4) allows for $\Delta_\top(E_1) \nvdash K$ and $\Delta_\top(E_1) \nvdash \neg K$.

From (KP2), we have that $\Delta_\top(E_2) \vdash K$ and thus $\Delta_\top(E_1) \wedge \Delta_\top(E_2)$ is consistent and from (KP6) we obtain $\Delta_\top(E_1 \sqcup E_2) \vdash \Delta_\top(E_1) \wedge \Delta_\top(E_2)$, i.e., $\Delta_\top(E_1) \vdash \Delta_\top(E_1) \wedge K$ and thus $\Delta_\top(E_1) \vdash K$ contradicting (KP4).

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Link between (IM) and (Maj)

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Theorem

If a merging operator satisfies (KP1) and (KP2) then it can not satisfies (IM) and (Maj) at the same time.

Proof

From (IM) and (Maj), we have for all E_1, K that $\Delta_\top(E_1 \sqcup K) \leftrightarrow \Delta_\top(E_1 \sqcup K^n) \vdash \Delta_\top(K)$.
From (KP2), we deduce that $\forall K, \Delta_\top(E_1 \sqcup K) \vdash K$.

Consider K' such that $K \wedge K' \vdash \bot$. Then with $E = K'$, we have $\Delta_\top(K' \sqcup K) \vdash K$. And also that $\Delta_\top(K \sqcup K') \vdash K'$ and thus that $\Delta_\top(K' \sqcup K) \vdash K \wedge K'$. Finally, $\Delta_\top(K' \sqcup K) \vdash \bot$ contradicting (KP1).

Link between (IM) and (Maj)

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Theorem

If a merging operator satisfies (KP1) and (KP2) then it can not satisfies (IM) and (Maj) at the same time.

Proof

From (IM) and (Maj), we have for all E_1, K that
$\Delta_\top(E_1 \sqcup K) \leftrightarrow \Delta_\top(E_1 \sqcup K^n) \vdash \Delta_\top(K)$.
From (KP2), we deduce that $\forall K, \Delta_\top(E_1 \sqcup K) \vdash K$.

Consider K' such that $K \wedge K' \vdash \bot$. Then with $E = K'$, we have $\Delta_\top(K' \sqcup K) \vdash K$. And also that $\Delta_\top(K \sqcup K') \vdash K'$ and thus that $\Delta_\top(K' \sqcup K) \vdash K \wedge K'$. Finally, $\Delta_\top(K' \sqcup K) \vdash \bot$ contradicting (KP1).

Syncretic assignment

Definition

A syncretic assignment is a function which associates to a belief profile E a pre-order $\leq_E$ over the interpretations such that for every belief profile E, E_1, E_2 and every belief base K, K' the following conditions hold:

1. If $\omega \models E$ and $\omega' \models E$ then $\omega \simeq_E \omega'$
2. If $\omega \models E$ and $\omega' \not\models E$ then $\omega <_E \omega'$
3. If $E_1 \leftrightarrow E_2$ then $\leq_{E_1} = \leq_{E_2}$
4. $\forall \omega \models K, \exists \omega' \models K', \omega' \leq_{K \sqcup K'} \omega$
5. If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$ then $\omega \leq_{E_1 \sqcup E_2} \omega'$
6. If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$ then $\omega <_{E_1 \sqcup E_2} \omega'$

Syncretic assignment - Extra conditions

Definition

A majority syncretic assignment is a syncretic assignment which satisfies the following condition:

7 If $\omega <_{E_2} \omega'$, then $\exists n, \omega <_{E_1 \sqcup E_2^n} \omega'$

Definition

A fair syncretic assignment is a syncretic assignment which satisfies the following condition:

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$$\left.\begin{array}{r} \omega <_K \omega' \\ \omega <_{K'} \omega'' \\ \omega' \simeq_{K \sqcup K'} \omega'' \end{array}\right\} \Rightarrow \omega <_{K \sqcup K'} \omega'$$

Syncretic assignment and KP postulates

Theorem

We consider Δ_{IC} a merging operation. Δ_{IC} respects all (KP) postulates iff there exists a syncretic assignment which associates to every belief profile E a total pre-order $\leq_E$ such that the result of the merging operation $\Delta_{IC}(E)$ as the set of minimal elements of $Mod(IC)$ according to the pre-order $\leq_E$.

Theorem

An operator Δ is a majority (resp. arbitration) merging operation iff there exists a majority (resp. fair) syncretic assignment which associates to every belief profile E a total pre-order $\leq_E$ such that the result of the merging operation $\Delta_{IC}(E)$ as the set of minimal elements of $Mod(IC)$ according to the pre-order $\leq_E$.

Distances and aggregation functions

Definition

Distances

$d : \Omega \times \Omega \to \mathbb{N}$ is a distance between interpretations iff it respects

1. $\forall \omega_1, \omega_2 \in \Omega, d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$
2. $d(\omega_1, \omega_2) = 0$ iff $\omega_1 = \omega_2$

It induces the distance between an interpretation and a formula:

$$d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')$$

Aggregation function

$f : \mathbb{N}^n \to \mathbb{N}$ is an aggregation function iff it respects

1. f is non-decreasing in each argument;
2. $\forall (x_1, \dots, x_n), f(x_1, \dots, x_n) = 0$ iff $x_1 = \dots = x_n = 0$;
3. $\forall x_1, f(x_1) = x_1$

Distances and aggregation functions

Example

Some distance functions:

drastic $d_D(\omega_1, \omega_2) = 0$ if $\omega_1 = \omega_1$, 1 otherwise

Hamming $d_H(\omega_1, \omega_2) = |\{x \in \mathcal{L} \mid \omega_1(x) \neq \omega_2(x)\}|$

Some aggregation functions: max, sum and lex.

Lexicographic aggregation

Given two vectors of numbers $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. Let σ and σ' be two permutations on $\{1, \ldots, n\}$ s.t. $\forall i, a_{\sigma(i)} \geq a_{\sigma(i+1)}$ and $b_{\sigma'(i)} \geq b_{\sigma'(i+1)}$.

$\vec{a} \leq_{lex} \vec{b}$ iff $\forall i, a_{\sigma(i)} = b_{\sigma'(i)}$ or $\exists i \geq 1$ s.t. $a_{\sigma(i)} < b_{\sigma'(i)}$ and $a_{\sigma(j)} = b_{\sigma'(j)}$ for all $1 \leq j < i$.

Distance-based merging operators

d is a distance, f and g are aggregation functions, $E = \{K_1, \ldots, K_n\}$ is belief profile and C is a formula:

$$\mathrm{Mod}(\Delta_{IC}^{d,f,g}(E)) = \{\omega \in \mathrm{Mod}(IC) \mid d(\omega, E) \text{ is minimal } \}$$

where

$$d(\omega, E) = g(d(\omega, K_1), \ldots, d(\omega, K_n))$$

and for every $K_i = \{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$

$$d(\omega, K_i) = f(d(\omega, \varphi_{i,1}), \ldots, d(\omega, \varphi_{i,n_i}))$$

Distance-based merging: example

Example

$E = \{K_1, K_2, K_3, K_4\}$ under the integrity constraint $IC = \top$ where

$$K_1 = \{a \wedge b \wedge c, a \rightarrow \neg b\}$$
$$K_2 = \{a \wedge b\}$$
$$K_3 = \{\neg a \wedge \neg b, \neg b\}$$
$$K_4 = \{a, a \rightarrow b\}$$

$\Delta^{d_H,\text{sum},\text{lex}}$ Operator.

	$a \wedge b \wedge c$	$a \rightarrow \neg b$	$a \vee b$	$\neg a \wedge \neg b$	$\neg b$	a	$a \rightarrow b$	K_1, K_2, K_3, K_4	E
000	3	0	2	0	0	1	0	3,2,0,1	3210
001	2	0	2	0	0	1	0	2,2,0,1	2210
010	2	0	1	1	1	1	0	2,1,2,1	2211
011	1	0	1	1	1	1	0	1,1,2,1	2111
100	2	0	1	1	0	0	1	2,1,1,1	2111
101	1	0	1	1	0	0	1	1,1,1,1	1111
110	1	1	0	2	1	0	0	2,0,3,0	3200
111	0	1	0	2	1	0	0	1,0,3,0	3100

Table of complexity

Complexity for d_D

f/g	max	sum	lex
max	BH_2	Θ_2^p	Θ_2^p
sum	Θ_2^p	Θ_2^p	Δ_2^p

Complexity for d_H

f/g	max	sum	lex
max	Θ_2^p	Θ_2^p	Δ_2^p
sum	Θ_2^p	Θ_2^p	Δ_2^p

Removed Sets Fusion: Principle

3 steps :

- subset of formulas which restore consistency: Potential Removed Sets
- minimal subset of formulas which restore consistency: Removed Sets
- profile without these formulas: Removed Sets Fusion operation

Potential Removed Set

$E = \{K_1, \ldots, K_n\}$: a belief profile $\quad IC$: constraints
s.t. $K_1 \sqcup \cdots \sqcup K_n \sqcup IC$ is inconsistent.
X : a subset of formulas from $K_1 \sqcup \cdots \sqcup K_n$.

Definition (Potential Removed Set)

X is a potential Removed Set of E constrainted by IC iff $((K_1 \sqcup \cdots \sqcup K_n) \backslash X) \sqcup IC$ is consistent.

Potential Removed Sets

$$K_1 = \{a \quad b\} \quad K_2 = \{\neg a \vee \neg b\}$$

Example

Potential Removed Sets	*Consistent subset*
$R_1 = \{a\}$	$E \backslash R_1 = \{\neg a \vee \neg b \quad b\}$
$R_2 = \{b\}$	$E \backslash R_2 = \{\neg a \vee \neg b \quad a\}$
$R_3 = \{\neg a \vee \neg b\}$	$E \backslash R_3 = \{a \quad b\}$
$R_4 = \{a \quad b\}$	$E \backslash R_4 = \{\neg a \vee \neg b\}$
$R_5 = \{b \quad \neg a \vee \neg b\}$	$E \backslash R_5 = \{a\}$
$R_6 = \{\neg a \vee \neg b \quad a\}$	$E \backslash R_6 = \{b\}$
$R_7 = \{\neg a \vee \neg b \quad a \quad b\}$	$E \backslash R_7 = \emptyset$

Removed Sets according to P

$E = \{K_1, \ldots, K_n\}$: a belief profile IC : constraints
s.t. $K_1 \sqcup \cdots \sqcup K_n \sqcup IC$ is inconsistent.
P : a merging strategy.

Definition (Removed Set)

X is a Removed Set of E constrained by IC according to P iff :

- X is a potential Removed Set of E constrained by IC;
- $\nexists X' \subseteq K_1 \sqcup \cdots \sqcup K_n$ s.t. $X' \subset X$;
- $\nexists X' \subseteq K_1 \sqcup \cdots \sqcup K_n$ s.t. $X' <_P X$.

Removed Sets

$$K_1 = \{a \quad b\} \quad K_2 = \{\neg a \vee \neg b\}$$

Example

Removed Sets	*Consistent subset*
$R_1 = \{a\}$	$E \backslash R_1 = \{\neg a \vee \neg b \quad b\}$
$R_2 = \{b\}$	$E \backslash R_2 = \{\neg a \vee \neg b \quad a\}$
$R_3 = \{\neg a \vee \neg b\}$	$E \backslash R_3 = \{a \quad b\}$
$R_4 = \{a \quad b\}$	$E \backslash R_4 = \{\neg a \vee \neg b\}$
$R_5 = \{b \quad \neg a \vee \neg b\}$	$E \backslash R_5 = \{a\}$
$R_6 = \{\neg a \vee \neg b \quad a\}$	$E \backslash R_6 = \{b\}$
$R_7 = \{\neg a \vee \neg b \quad a \quad b\}$	$E \backslash R_7 = \emptyset$

Definition of the merging operator

$E = \{K_1, \ldots, K_n\}$: a belief profile $\quad IC$: constraints
P : a merging strategy.
$\mathcal{F}_{P,IC}\mathcal{R}(E)$: the set of Removed Sets of E constrainted by IC according to P.

Definition ($\Delta^{RSF}_{P,IC}(E)$)

$$\Delta^{RSF}_{P,IC}(E) = \bigvee_{X \in \mathcal{F}_{P,IC}\mathcal{R}(E)} \{((K_1 \sqcup \cdots \sqcup K_n) \backslash X) \sqcup IC\}$$

Example

$$K_1 = \{a \quad b\} \quad K_2 = \{\neg a \vee \neg b\}$$
$$\Delta^{RSF}_{\Sigma,IC}(E) = \{\neg a \vee \neg b \quad b\} \vee \{\neg a \vee \neg b \quad a\} \vee \{a \quad b\}$$

Pre-order *Sum*

$E = \{K_1, \ldots, K_n\}$: a belief profile.
X, X' : two potential Removed Sets of E.

Definition ($\leq_\Sigma$)

$$X \leq_\Sigma X' \text{ iff } \sum_{1 \leq i \leq n} |X \cap K_i| \leq \sum_{1 \leq i \leq n} |X' \cap K_i|$$

The *Sum* strategy

Profile $E = \{K_1, K_2, K_3\}$

$$K_1 = \{\neg d,\quad s \vee o,\quad s\} \qquad K_2 = \{\neg s,\quad d \vee o,\quad \neg d \vee \neg o\}$$

$$K_3 = \{s,\quad d,\quad o\}$$

6	$s \vee o \quad s, \quad d \vee o, \quad s, \quad d, \quad o$
5	$s \vee o, \quad \neg d, \quad s, \quad s, \quad o$
4	$\neg d, \quad s, \quad \neg d \vee \neg o, \quad s$
—	$d \vee o, \quad \neg s, \quad d, \quad o$
3	$s, \quad s, \quad d$
—	$\neg d, \quad \neg s, \quad o$
—	$\neg d, \quad \neg s, \quad \neg d \vee \neg o$
2	$\neg s, \quad d$

$$\Delta^{RSF}_{\Sigma, IC}(E) = \{\neg d \quad s \vee o \quad s \quad d \vee o \quad \neg d \vee \neg o \quad s \quad o\}$$

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