Principles of
Knowledge Representation and Reasoning
Dynamics of belief

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1 Introduction

- Link between revision and update
Principles

Propositional logic flaws:

- The world is not always static.
- The knowledge about the world is sometimes uncertain or imprecise

Therefore:

- Need the possibility to incorporate new (possibly contradictory) beliefs;
- Need to take into account change in the world;
Plato - Theaetetus: A knowledge (a rightful opinion) is a piece of

1. Justified True Belief

Agrippa’s trilemma - A problem with the justification:

1. Either the justification stops to some unjustified belief;
2. The justification is infinite (Socrates’ clouds);
3. The justification is supported by affirmations it is supposed to justify (Baron Münchhausen’s hair).
Foundationalism and coherentism

Three solutions:

Foundationalism  Allow for unjustified beliefs
   → Formalization issues
   → Humans don’t keep track of sources
   → TMS System

“Infinitism”  Allow for infinite justification
   → Does it really make sense?

Coherentism  Allow for circular justifications
   → What is a solid belief?
   → Belief revision/update

► In any cases, information is extremely important and should not be discarded carelessly.
Social choice theory: the Arrow theorem

Arrow’s impossibility theorem - there is no voting system which respects:

- Non-dictatorship
  (all voters should be taken into account);
- Universality
  (complete and deterministic ranking);
- Independence of irrelevant alternatives
  (ranking between \( x \) and \( y \) depends only on \( x \) and \( y \));
- Pareto efficiency
  (if all preferences states \( x < y \), then so must the results).
Revision or update

- We have a theory about the world, and the new information is meant to **correct** our theory

  belief revision: change your belief state minimally in order to accommodate the new information

- We have a (supposedly) correct theory about the current state of the world, and the new information is meant to record a **change** in the world

  belief update: incorporate the change by assuming that the world has changed minimally
Assume the new information is consistent with our old beliefs.

- In case of belief revision, we would like to add the new information monotonically to our old beliefs.
- For belief update this is not necessarily the case.
  - Assume we know that the door is open or the window is open.
  - Assume we learn that the world has changed and the door is now closed.
- In this case, we do not want to add this information monotonically to our theory, since we would be forced to conclude that the window is open.
Overview of an operation

What are the criteria for definition of a belief revision operation?

Gärdenfors and Rott - belief revision (1995):

1. How are beliefs represented?
2. What is the relation between beliefs represented explicitly in the belief base and beliefs which can be derived from them?
3. In the face of a contradiction, how to deal with both new and old information?
Belief base, belief set or interpretation?

General assumption:

- A belief set is a deductively closed theory, i.e., $K = \text{Cn}(K)$ with Cn the consequence operator.
- $\mathcal{L}$: logical language (propositional logic).
- $\text{Th}_{\mathcal{L}}$: set of deductively closed theories (or belief sets) over $\mathcal{L}$.

Belief change operations

Monotonic addition: $+: \text{Th}_{\mathcal{L}} \times \mathcal{L} \rightarrow \text{Th}_{\mathcal{L}}$

$$K + \psi = \text{Cn}(K \cup \{\psi\})$$

Revision: $\dot{+}: \text{Th}_{\mathcal{L}} \times \mathcal{L} \rightarrow \text{Th}_{\mathcal{L}}$
Consider $K = \{a, b\}$ and $K' = \{a \land b\}$. What is happening to $K + \{\neg a\}$?

**Semantic**

- No difference between $K$ and $K'$

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**Syntactic**

- $X = \{b\}$ is the only maximal subset of $K$ s.t. $X \cup \{\neg a\}$ is consistent.

- $X' = \emptyset$ is the only maximal subset of $K'$ s.t. $X' \cup \{\neg a\}$ is consistent.
2 Belief revision

- Formal properties
- Standard revision operations
- Semantic approaches
What is a good revision operator?

- Consistency: a revision has to produce a consistent set of beliefs;
- Minimality of change: a revision has to change the fewest possible beliefs;
- Priority to the new information: the ’new’ information is considered more important than the ’old’ one.
The AGM postulates
Characterization for belief sets’ revision

AGM postulates:

(\dagger 1) \( K \vdash \varphi \in \text{Th}_L \); 
(\dagger 2) \( \varphi \in K \vdash \varphi \); 
(\dagger 3) \( K \vdash \varphi \subseteq K + \varphi \); 
(\dagger 4) If \( \neg \varphi \not\in K \), then \( K + \varphi \subseteq K \vdash \varphi \); 
(\dagger 5) \( K \vdash \varphi = \text{Cn}(\bot) \) only if \( \vdash \neg \varphi \); 
(\dagger 6) If \( \vdash \varphi \leftrightarrow \psi \) then \( K \vdash \varphi = K \vdash \psi \); 
(\dagger 7) \( K \vdash (\varphi \land \psi) \subseteq (K \vdash \varphi) + \psi \); 
(\dagger 8) If \( \neg \psi \not\in K \vdash \varphi \), 
then \( (K \vdash \varphi) + \psi \subseteq K \vdash (\varphi \land \psi) \).
The Levi identity

Revision can be defined in terms of two suboperations.
- + (expansion) denotes the simple union of beliefs;
- − (contraction) denotes the removal of information contradicting the input.

The Levi identity

\[ K \dot{+} \varphi \equiv Cn[(K - \neg \varphi) + \varphi] \]

Example

\[ K = \{a, a \rightarrow b\} \quad \varphi\{\neg b\}? \]

\[ K - \neg \varphi = \{a\} \text{ or } \{a \rightarrow b\} \]

\[ K \dot{+} \neg \varphi = \{a, \neg b\} \text{ or } \{a \rightarrow b, \neg b\} \]
Full-meet contraction

**Definition**

We denote by $K \perp \varphi$ the set of maximal (wrt set-theoretic inclusion) subsets $J$ of $K$ such that $J \not\models \varphi$.

**Definition**

Full-meet contraction is defined by $K - \varphi = \bigcap (K \perp \varphi)$.

Is full-meet contraction reasonable?

- No! It is far too cautious.
- It can nevertheless be used as a lower bound to any reasonable operator.

$K \dot{+} \varphi = \bigcap (K \perp \varphi) + \varphi$ is referred to as the full-meet revision.
Full-meet contraction

Properties

Proposition

*Full-meet revision respects all AGM postulates.*

Proof

(\(\dagger\)) and (\(\dagger\)2) are true by construction

(\(\dagger\)3) Two cases: (1) If \(K + \varphi\) is consistent then \(K - \varphi = K\) and \(K + \varphi = K + \varphi\). (2) If \(K + \varphi\) is inconsistent then \(K + \varphi = \text{Cn}(\bot)\) and \(K + \varphi \subseteq K + \varphi\).

(\(\dagger\)4) Because \(K \not\vdash \neg \varphi\) then \(K \bot \varphi = \{K\}\) and thus \(K + \varphi = K + \varphi\).

(\(\dagger\)5) \(K + \varphi = \text{Cn}\left( \bigcap_{\alpha \in (K \bot \varphi)} \alpha \cup \varphi \right)\). But \(\forall \alpha, \alpha \cup \varphi \not\vdash \bot\), therefore \(\bigcap_{\alpha \in (K \bot \varphi)} \alpha \cup \varphi \not\vdash \bot\) (as PL is monotonic).

(\(\dagger\)6) Lets assume that \(\alpha \in K \bot \varphi\) but \(\alpha \not\subseteq K \bot \Psi\). Two cases: (1) \(\alpha \cup \Psi \vdash \bot \Rightarrow (\varphi \iff \Psi)\) \(\Rightarrow \alpha \cup \varphi \vdash \bot\) which is not possible. (2) \(\exists \beta\) s.t. \(\alpha \not\subseteq \beta\) and \(\beta \cup \Psi \not\vdash \bot\) \(\Rightarrow (\varphi \iff \Psi)\) \(\Rightarrow \beta \cup \varphi \not\vdash \bot\) which is not possible.

(\(\dagger\)7) and (\(\dagger\)8) Left as exercises...
Maxi-choice contraction

On the other side, one can ask for the principle of minimality to be strictly respected.

**Definition**

A selection function for $K$ is a function $\gamma$ such that for all sentences $\varphi$:

1. If $K \perp \varphi$ is non-empty, then $\gamma(K \perp \varphi)$ is a non-empty subset of $K \perp \varphi$, and
2. If $K \perp \varphi$ is empty, then $\gamma(K \perp \varphi) = \{K\}$.

**Definition**

Maxichoice contraction is defined as $K - \varphi = \gamma(K \perp \varphi)$ where $\gamma$ is a selection function.
Partial-meet contraction

Maxi-choice can be too bold: there is sometimes no reason to trust one piece more than one another.

Definition
A partial-meet revision operation is an operation defined as:

\[ K \oplus \varphi = \bigcap \gamma(K \perp \varphi) + \varphi \]

Seems to be a good compromise between full-meet and maxi-choice
**Definition**

The Dalal revision operation, denoted by $\hat{\oplus}_D$, is defined as:

$$K \hat{\oplus}_D \varphi = \min(\text{extMod}(\varphi), \leq_K)$$

where $d_H$ is the Hamming Distance and

$$\alpha \leq_K \beta \iff \exists \omega \in \text{extMod}(K), \forall \omega' \in \text{extMod}(K), d_H(\alpha, \omega) \leq d_H(\beta, \omega')$$

**Example**

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Let $\varphi = \{\neg a, \neg b\}$ and $K = \{(a \lor b) \land c\}$:

$$d(I_{\varphi_1}, I_{K_1}) = 2 \quad d(I_{\varphi_2}, I_{K_1}) = 1$$
$$d(I_{\varphi_1}, I_{K_2}) = 2 \quad d(I_{\varphi_2}, I_{K_2}) = 1$$
$$d(I_{\varphi_1}, I_{K_3}) = 3 \quad d(I_{\varphi_2}, I_{K_3}) = 2$$
Some complexity result

Formula-based approaches

The question does $\Psi$ belongs to $K \vdash \varphi$ (if $\vdash$ is a full-meet revision operator) is $\Delta^p_2 - (\Sigma^p_1 \cup \Pi^p_1)$ provided that NP $\neq$ co-NP.

proof

If $\vdash$ is a full-meet revision, $\Psi \in \text{Cn}(K) \vdash \varphi$ can be solved by the following algorithm: if $K \not\models \neg \Psi$, then $K \cup \Psi \models \varphi$ else $\Psi \models \varphi \rightarrow$ Membership in $\Delta^p_2$.

Furthermore, SAT can be polynomially transformed to full-meet revision by solving $\Psi \in \text{Cn}(\Psi) \vdash \top$ and UNSAT can be polynomially transform to full-meet revision by solving $\bot \in \text{Cn}(\emptyset) \vdash \Psi$. Hence, assuming that full-meet revision belongs to both NP and co-NP would lead to NP = co-NP.
3 Several sources - belief merging

- Postulational aspects
- Distance-based merging
- Syntactic merging
There is not only one source for the information:

- Voting procedure;
- Expert system;
- Distributed databases;
- multisource knowledge acquisition.

Constructing a belief base which represents the several sources and which:

- solves the contradiction;
- reduces the redundancies;
- is consistent.
Merging in the general case

\[ E = \{K_1, K_2, \ldots, K_n\} \]

Each \( K_i \) is consistent

\( K_1 \sqcup K_2 \sqcup \cdots \sqcup K_n \) is inconsistent
Formal framework

General assumption:

- $K_1, \ldots, K_n$ are belief bases;
- $E = \{K_1, \ldots, K_n\}$ is a multi-set of belief bases and is called a belief profile;
- $IC$ is a propositional formula standing for constraints;
- $\square$ stands for multi-set union.

Operation

Belief merging operation: $\Delta : \mathcal{L}^n \times \mathcal{L} \to \mathcal{L}$

Sometimes also called fusion operation.
Konieczny-PinoPerez postulates

(KP0) \( \Delta_{IC}(E) \models IC \).

(KP1) If \( IC \) is consistent, then \( \Delta_{IC}(E) \) is consistent.

(KP2) If \( \bigwedge E \land IC \) is consistent, then \( \Delta_{IC}(E) = \bigwedge E \land IC \).

(KP3) If \( E_1 \equiv E_2 \) and \( IC_1 \equiv IC_2 \), then
\[
\Delta_{IC_1}(E_1) \equiv \Delta_{IC_2}(E_2).
\]

(KP4) If \( K_1 \models IC \) and \( K_2 \models IC \), then
\[
\Delta_{IC}(K_1 \sqcup K_2) \land K_1 \not\models \bot \text{ implies }
\Delta_{IC}(K_1 \sqcup K_2) \land K_2 \not\models \bot.
\]

(KP5) \( \Delta_{IC}(E_1) \land \Delta_{IC}(E_2) \models \Delta_{IC}(E_1 \sqcup E_2) \).

(KP6) If \( \Delta_{IC}(E_1) \land \Delta_{IC}(E_2) \) is consistent, then
\[
\Delta_{IC}(E_1 \sqcup E_2) \models \Delta_{IC}(E_1) \land \Delta_{IC}(E_2).
\]

(KP7) \( \Delta_{IC_1}(E) \land IC_2 \models \Delta_{IC_1 \land IC_2}(E) \).

(KP8) If \( \Delta_{IC_1}(E) \land IC_2 \) is consistent, then
\[
\Delta_{IC_1 \land IC_2}(E) \models \Delta_{IC_1}(E) \land IC_2.
\]
Arbitration or majority operations

Arbitration (Arb)

\[
\begin{align*}
\Delta_{IC_1}(K_1) & \leftrightarrow \Delta_{IC_2}(K_2) \\
\Delta_{IC_1} & \leftrightarrow \lnot IC_2(K_1 \sqcup K_2) \leftrightarrow (IC_1 & \leftrightarrow \lnot IC_2) \\
IC_1 & \not\vdash IC_2 \\
IC_2 & \not\vdash IC_1
\end{align*}
\]

\[\Rightarrow \Delta_{IC_1 \lor IC_2}(K_1 \sqcup K_2) \leftrightarrow \Delta_{IC_1}(K_1)\]

Majority (Maj)

\[
\exists n, \Delta_{IC}(K_1 \sqcup K_2^n) \vdash \Delta_{IC}(K_2)
\]

Independence from majority (IM)

\[
\forall n, \Delta_{IC}(K_1 \sqcup K_2^n) \leftrightarrow \Delta_{IC}(K_1 \sqcup K_2)
\]
Link between (IM) the KP postulates

Theorem

There exists no merging operator satisfying all the KP postulates and (IM).

Proof

Consider $E_1 = \{K, \neg K\}$ and $E_2 = \{K\}$ be two belief profiles. (IM) leads to $\Delta_T(E_1 \sqcup E_2) = \Delta_T(E_1)$. (KP4) allows for $\Delta_T(E_1) \not\vdash K$ and $\Delta_T(E_1) \not\vdash \neg K$. From (KP2), we have that $\Delta_T(E_2) \vdash K$ and thus $\Delta_T(E_1) \wedge \Delta_T(E_2)$ is consistent and from (KP6) we obtain $\Delta_T(E_1 \sqcup E_2) \vdash \Delta_T(E_1) \wedge \Delta_T(E_2)$, i.e., $\Delta_T(E_1) \vdash \Delta_T(E_1) \wedge K$ and thus $\Delta_T(E_1) \vdash K$ contradicting (KP4).
Link between (IM) and (Maj)

Theorem

*If a merging operator satisfies (KP1) and (KP2) then it can not satisfies (IM) and (Maj) at the same time.*

Proof

From (IM) and (Maj), we have for all $E_1, K$ that

$$\Delta_T(E_1 \sqcup K) \leftrightarrow \Delta_T(E_1 \sqcup K^n) \vdash \Delta_T(K).$$

From (KP2), we deduce that $\forall K, \Delta_T(E_1 \sqcup K) \vdash K$.

Consider $K'$ such that $K \land K' \vdash \bot$. Then with $E = K'$, we have

$$\Delta_T(K' \sqcup K) \vdash K.$$ And also that $\Delta_T(K \sqcup K') \vdash K'$ and thus that $\Delta_T(K' \sqcup K) \vdash K \land K'$. Finally, $\Delta_T(K' \sqcup K) \vdash \bot$ contradicting (KP1).
Syncretic assignment

Definition

A syncretic assignment is a function which associates to a belief profile $E$ a pre-order $\leq_E$ over the interpretations such that for every belief profile $E, E_1, E_2$ and every belief base $K, K'$ the following conditions hold:

1. If $\omega \models E$ and $\omega' \models E$ then $\omega \simeq_E \omega'$
2. If $\omega \models E$ and $\omega' \not\models E$ then $\omega <_E \omega'$
3. If $E_1 \leftrightarrow E_2$ then $\leq_{E_1} = \leq_{E_2}$
4. $\forall \omega \models K, \exists \omega' \models K', \omega' \leq_{K \sqcup K'} \omega$
5. If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$ then $\omega \leq_{E_1 \cup E_2} \omega'$
6. If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$ then $\omega <_{E_1 \cup E_2} \omega'$
Syncretic assignment - Extra conditions

Definition
A majority syncretic assignment is a syncretic assignment which satisfies the following condition:

7 \[ \text{If } \omega <_{E_2} \omega', \text{ then } \exists n, \omega <_{E_1 \sqcup E_2^n} \omega' \]

Definition
A fair syncretic assignment is a syncretic assignment which satisfies the following condition:

8 \[ \begin{align*}
\omega &<_{K} \omega' \\
\omega &<_{K'} \omega'' \\
\omega' &\simeq_{K \sqcup K'} \omega''
\end{align*} \Rightarrow \omega <_{K \sqcup K'} \omega' \]
Syncretic assignment and KP postulates

Theorem

We consider $\Delta_{IC}$ a merging operation. $\Delta_{IC}$ respects all (KP) postulates iff there exists a syncretic assignment which associates to every belief profile $E$ a total pre-order $\leq_E$ such that the result of the merging operation $\Delta_{IC}(E)$ as the set of minimal elements of $\text{Mod}(IC)$ according to the pre-order $\leq_E$.

Theorem

An operator $\Delta$ is a majority (resp. arbitration) merging operation iff there exists a majority (resp. fair) syncretic assignment which associates to every belief profile $E$ a total pre-order $\leq_E$ such that the result of the merging operation $\Delta_{IC}(E)$ as the set of minimal elements of $\text{Mod}(IC)$ according to the pre-order $\leq_E$. 
Distances and aggregation functions

Definition

Distances

\( d : \Omega \times \Omega \rightarrow \mathbb{N} \) is a distance between interpretations iff it respects

1. \( \forall \omega_1, \omega_2 \in \Omega, d(\omega_1, \omega_2) = d(\omega_2, \omega_1) \)
2. \( d(\omega_1, \omega_2) = 0 \) iff \( \omega_1 = \omega_2 \)

It induces the distance between an interpretation and a formula:

\[
    d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')
\]

Aggregation function

\( f : \mathbb{N}^n \rightarrow \mathbb{N} \) is an aggregation function iff it respects

1. \( f \) is non-decreasing in each argument;
2. \( \forall (x_1, \ldots, x_n), f(x_1, \ldots, x_n) = 0 \) iff \( x_1 = \ldots = x_n = 0 \);
3. \( \forall x_1, f(x_1) = x_1 \)
Distances and aggregation functions

Example

Some distance functions:

- **drastic**  $d_D(\omega_1, \omega_2) = 0$ if $\omega_1 = \omega_1$, $1$ otherwise
- **Hamming**  $d_H(\omega_1, \omega_2) = |\{x \in \mathcal{L} \mid \omega_1(x) \neq \omega_2(x)\}|$

Some aggregation functions: max, sum and lex.

**Lexicographic aggregation**

Given two vectors of numbers $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. Let $\sigma$ and $\sigma'$ be two permutations on $\{1, \ldots, n\}$ s.t. $\forall i, a_{\sigma}(i) \geq a_{\sigma}(i+1)$ and $b_{\sigma'}(i) \geq b_{\sigma'}(i+1)$.

$\vec{a} \leq_{lex} \vec{b}$ iff $\forall i, a_{\sigma}(i) = b_{\sigma'}(i)$ or $\exists i \geq 1$ s.t. $a_{\sigma}(i) < b_{\sigma'}(i)$ and $a_{\sigma}(j) = b_{\sigma'}(j)$ for all $1 \leq j < i$. 
Distance-based merging operators

\( d \) is a distance, \( f \) and \( g \) are aggregation functions,
\( E = \{K_1, \ldots, K_n\} \) is belief profile and \( C \) is a formula:

\[
\text{Mod}(\Delta_{IC}^{d,f,g}(E)) = \{ \omega \in \text{Mod}(IC) \mid d(\omega, E) \text{ is minimal} \}
\]

where

\[
d(\omega, E) = g(d(\omega, K_1), \ldots, d(\omega, K_n))
\]

and for every \( K_i = \{\phi_{i,1}, \ldots, \phi_{i,n_i}\} \)

\[
d(\omega, K_i) = f(d(\omega, \phi_{i,1}), \ldots, d(\omega, \phi_{i,n_i}))
\]
Distance-based merging: example

Example

\[ E = \{K_1, K_2, K_3, K_4\} \text{ under the integrity constraint } IC = \top \text{ where} \]

\[
K_1 = \{a \land b \land c, a \rightarrow \neg b\} \\
K_2 = \{a \land b\} \\
K_3 = \{\neg a \land \neg b, \neg b\} \\
K_4 = \{a, a \rightarrow b\}
\]

\[ \Delta_{d_H, \text{sum,lex}} \text{ Operator.} \]

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#### Complexity for $d_D$

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<td>$\Delta_p^2$</td>
</tr>
</tbody>
</table>

#### Complexity for $d_H$

<table>
<thead>
<tr>
<th>$f/g$</th>
<th>max</th>
<th>sum</th>
<th>lex</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>$\Theta_p^2$</td>
<td>$\Theta_p^2$</td>
<td>$\Delta_p^2$</td>
</tr>
<tr>
<td>sum</td>
<td>$\Theta_p^2$</td>
<td>$\Theta_p^2$</td>
<td>$\Delta_p^2$</td>
</tr>
</tbody>
</table>
Removed Sets Fusion: Principle

3 steps:

- subset of formulas which restore consistency: Potential Removed Sets
- minimal subset of formulas which restore consistency: Removed Sets
- profile without these formulas: Removed Sets Fusion operation
Potential Removed Set

\[ E = \{K_1, \ldots, K_n\} : \text{a belief profile} \quad IC : \text{constraints} \]
\[ \text{s.t. } K_1 \sqcup \cdots \sqcup K_n \sqcup IC \text{ is inconsistent.} \]
\[ X : \text{a subset of formulas from } K_1 \sqcup \cdots \sqcup K_n. \]

Definition (Potential Removed Set)

\[ X \text{ is a potential Removed Set of } E \text{ constrained by } IC \text{ iff} \]
\[ ((K_1 \sqcup \cdots \sqcup K_n) \setminus X) \sqcup IC \text{ is consistent.} \]
Potential Removed Sets

\[ K_1 = \{a, b\} \quad K_2 = \{\neg a \lor \neg b\} \]

Example

Potential Removed Sets

\[ R_1 = \{a\} \]
\[ R_2 = \{b\} \]
\[ R_3 = \{\neg a \lor \neg b\} \]
\[ R_4 = \{a, b\} \]
\[ R_5 = \{b, \neg a \lor \neg b\} \]
\[ R_6 = \{\neg a \lor \neg b, a\} \]
\[ R_7 = \{\neg a \lor \neg b, a, b\} \]

Consistent subset

\[ E \backslash R_1 = \{\neg a \lor \neg b, b\} \]
\[ E \backslash R_2 = \{\neg a \lor \neg b, a\} \]
\[ E \backslash R_3 = \{a, b\} \]
\[ E \backslash R_4 = \{\neg a \lor \neg b\} \]
\[ E \backslash R_5 = \{a\} \]
\[ E \backslash R_6 = \{b\} \]
\[ E \backslash R_7 = \emptyset \]
Removed Sets according to $P$

$E = \{K_1, \ldots, K_n\}$: a belief profile  
$IC$ : constraints  
s.t. $K_1 \sqcup \cdots \sqcup K_n \sqcup IC$ is inconsistent.  
$P$ : a merging strategy.

Definition (Removed Set)

$X$ is a Removed Set of $E$ constrained by $IC$ according to $P$ iff:

- $X$ is a potential Removed Set of $E$ constrained by $IC$;
- $\forall X' \subseteq K_1 \sqcup \cdots \sqcup K_n$ s.t. $X' \subset X$;
- $\forall X' \subseteq K_1 \sqcup \cdots \sqcup K_n$ s.t. $X' <_P X$. 

Removed Sets

\[ K_1 = \{a, b\} \quad K_2 = \{\neg a \lor \neg b\} \]

Example

**Removed Sets**
- \( R_1 = \{a\} \)
- \( R_2 = \{b\} \)
- \( R_3 = \{\neg a \lor \neg b\} \)
- \( R_4 = \{a, b\} \)
- \( R_5 = \{b, \neg a \lor \neg b\} \)
- \( R_6 = \{\neg a \lor \neg b, a\} \)
- \( R_7 = \{\neg a \lor \neg b, a, b\} \)

**Consistent subset**
- \( E \setminus R_1 = \{\neg a \lor \neg b, b\} \)
- \( E \setminus R_2 = \{\neg a \lor \neg b, a\} \)
- \( E \setminus R_3 = \{a, b\} \)
- \( E \setminus R_4 = \{\neg a \lor \neg b\} \)
- \( E \setminus R_5 = \{a\} \)
- \( E \setminus R_6 = \{b\} \)
- \( E \setminus R_7 = \emptyset \)
**Definition of the merging operator**

\[ E = \{K_1, \ldots, K_n\} : \text{a belief profile} \quad IC : \text{constraints} \]

\[ P : \text{a merging strategy.} \]

\[ \mathcal{F}_{P,IC}(E) : \text{the set of Removed Sets of } E \text{ constrained by } IC \text{ according to } P. \]

**Definition (\(\Delta_{P,IC}^{RSF}(E)\))**

\[
\Delta_{P,IC}^{RSF}(E) = \bigvee_{X \in \mathcal{F}_{P,IC}(E)} \{((K_1 \sqcup \cdots \sqcup K_n) \setminus X) \cup IC\}
\]

**Example**

\[
K_1 = \{a, b\} \quad K_2 = \{\neg a \lor \neg b\}
\]

\[
\Delta_{\Sigma,IC}^{RSF}(E) = \{\neg a \lor \neg b\} \lor \{\neg a \lor \neg b, a\} \lor \{a, b\}
\]
Pre-order *Sum*

\[ E = \{K_1, \ldots, K_n\} : \text{a belief profile.} \]
\[ X, X' : \text{two potential Removed Sets of } E. \]

**Definition (≤Σ)**

\[ X \leq_{\Sigma} X' \iff \sum_{1 \leq i \leq n} |X \cap K_i| \leq \sum_{1 \leq i \leq n} |X' \cap K_i| \]
The Sum strategy

Profile $E = \{K_1, K_2, K_3\}$

$K_1 = \{\neg d, \ s \lor o, \ s\} \quad K_2 = \{\neg s, \ d \lor o, \ \neg d \lor \neg o\} \quad K_3 = \{s, \ d, \ o\}$

\[
\begin{array}{c|cccccc}
6 & s \lor o & s & d \lor o & s & d & o \\
5 & s \lor o & \neg d & s & s & o \\
4 & \neg d & s & \neg d \lor \neg o & s \\
3 & d \lor o & \neg s & d & o \\
2 & s & s & d \\
1 & \neg d & \neg s & o \\
0 & \neg d & \neg s & \neg d \lor \neg o \\
\end{array}
\]

$\Delta^{RSF}_{\Sigma,IC}(E) = \{\neg d \ s \lor o \ s \ d \lor o \ \neg d \lor \neg o \ s \ o\}$
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