Introduction
Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models).
- **Nonmonotonicity** is only a **negative** characterization: From $\Theta \not\models \varphi$, it does not necessarily follow $\Theta \cup \{\psi\} \not\models \varphi$.
- Could we have a constructive **positive** characterization of default reasoning?
### Plausible consequences

- In classical logics, we have the logical consequence relation $\alpha \models \beta$: If $\alpha$ is true, then also $\beta$ is true.
- Instead, we will study the relation of **plausible consequence** $\alpha \sim \beta$: If $\alpha$ is all we know, can we conclude $\beta$?
- $\alpha \sim \beta$ does not imply $\alpha \land \alpha' \sim \beta$!
  Compare to conditional probability: $P(\beta | \alpha) \neq P(\beta | \alpha, \alpha')$!
- Find rules that characterize $\sim$ ... 
  For example: if $\alpha \sim \beta$ and $\alpha \sim \gamma$, then $\alpha \sim \beta \land \gamma$.
- Write down all such rules ... 
- ... and find a semantic characterization of $\sim$!
Desirable properties: Reflexivity

Reflexivity (Ref):

\[ \alpha \sim \alpha \]

- **Rationale**: If \( \alpha \) holds, this *normally implies* \( \alpha \).
- **Example**: Tom goes to a party *normally implies* that Tom goes to a party.
Reflexivity in default logic

Let $\Delta = \langle D, W \rangle$ be a propositional default theory. Define the relation $\sim_\Delta$ as follows:

$$\alpha \sim_\Delta \beta \iff \langle D, W \cup \{\alpha\} \rangle \sim \beta$$

$\alpha \sim_\Delta \beta$ means that $\beta$ is a skeptical conclusion of $\langle D, W \cup \{\alpha\} \rangle$.

**Proposition**

Default logic satisfies Reflexivity.

**Proof.**

The question is: does $\alpha$ follow skeptically from $\Delta' = \langle D, W \cup \{\alpha\} \rangle$? For each extension $E$ of $\Delta'$, it holds $W \cup \{\alpha\} \subseteq E$ (by definition). Hence $\alpha \in E$, and thus $\alpha$ belongs to all extensions of $\Delta'$.
Reflexivity in default logic

Let $\Delta = \langle D, W \rangle$ be a propositional default theory. Define the relation $\not\models_\Delta$ as follows:

\[
\alpha \not\models_\Delta \beta \iff \langle D, W \cup \{\alpha\} \rangle \not\models \beta
\]

$\alpha \not\models_\Delta \beta$ means that $\beta$ is a skeptical conclusion of $\langle D, W \cup \{\alpha\} \rangle$.

**Proposition**

*Default logic satisfies Reflexivity.*

**Proof.**

The question is: does $\alpha$ follow skeptically from $\Delta' = \langle D, W \cup \{\alpha\} \rangle$? For each extension $E$ of $\Delta'$, it holds $W \cup \{\alpha\} \subseteq E$ (by definition). Hence $\alpha \in E$, and thus $\alpha$ belongs to all extensions of $\Delta'$.
Desirable properties: Left Logical Equivalence

**Left Logical Equivalence (LLE):**

\[ \models \alpha \leftrightarrow \beta, \ \alpha \not\models \gamma \]

\[ \beta \not\models \gamma \]

- **Rationale:** It is not the syntactic form, but the content that is responsible for what we conclude normally.

- **Example:** Assume that
  
  Tom goes or Peter goes normally implies Mary goes.

  Then we would expect that

  Peter goes or Tom goes normally implies Mary goes.
Proposition

Default logic satisfies Left Logical Equivalence.

Proof.

Assume $\models \alpha \iff \beta$ and $\alpha \Vdash_\Delta \gamma$ (with $\Delta = \langle D, W \rangle$).

Hence, $\gamma$ is in all extensions of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$.

The definition of extensions is invariant under replacing any formula by an equivalent formula.

Thus, $\langle D, W \cup \{\beta\} \rangle$ has exactly the same extensions as $\Delta'$, and $\gamma$ is in every one of them. Hence, $\beta \Vdash_\Delta \gamma$. 
Proposition

**Default logic satisfies Left Logical Equivalence.**

Proof.

Assume $\models \alpha \leftrightarrow \beta$ and $\alpha \mid\sim\Delta \gamma$ (with $\Delta = \langle D, W \rangle$).

Hence, $\gamma$ is in all extensions of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$.

The definition of extensions is invariant under replacing any formula by an equivalent formula.

Thus, $\langle D, W \cup \{\beta\} \rangle$ has exactly the same extensions as $\Delta'$, and $\gamma$ is in every one of them. Hence, $\beta \mid\sim\Delta \gamma$. 

\[\square\]
**Proposition**

*Default logic satisfies Left Logical Equivalence.*

**Proof.**

Assume \( \models \alpha \leftrightarrow \beta \) and \( \alpha \vdash_{\Delta} \gamma \) (with \( \Delta = \langle D, W \rangle \)).

Hence, \( \gamma \) is in all extensions of \( \Delta' := \langle D, W \cup \{\alpha\} \rangle \).

The definition of extensions is invariant under replacing any formula by an equivalent formula.

Thus, \( \langle D, W \cup \{\beta\} \rangle \) has exactly the same extensions as \( \Delta' \), and \( \gamma \) is in every one of them. Hence, \( \beta \vdash_{\Delta} \gamma \).
**Proposition**

*Default logic satisfies Left Logical Equivalence.*

**Proof.**

Assume $\models \alpha \leftrightarrow \beta$ and $\alpha \not\models_{\Delta} \gamma$ (with $\Delta = \langle D, W \rangle$). Hence, $\gamma$ is in all extensions of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$.

The definition of extensions is invariant under replacing any formula by an equivalent formula. Thus, $\langle D, W \cup \{\beta\} \rangle$ has exactly the same extensions as $\Delta'$, and $\gamma$ is in every one of them. Hence, $\beta \not\models_{\Delta} \gamma$.  

$\square$
Desirable properties: Right Weakening

Right Weakening (RW):

\[ \vdash \alpha \rightarrow \beta, \gamma \models \alpha \quad \Rightarrow \quad \gamma \models \beta \]

- **Rationale**: If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example**: Assume that
  Mary goes normally implies Clive goes and John goes.
  Then we would expect that
  Mary goes normally implies Clive goes.

- **From (Ref) & (RW) Supraclassicality follows**:

  \[ \alpha \models \alpha \quad + \quad \vdash \alpha \rightarrow \beta, \alpha \models \alpha \quad \Rightarrow \quad \alpha \models \beta \]
Desirable properties: Right Weakening

Right Weakening (RW):

\[
\models \alpha \to \beta, \gamma \not\models \alpha \\
\gamma \not\models \beta
\]

- **Rationale**: If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example**: Assume that
  Mary goes normally implies Clive goes and John goes. Then we would expect that
  Mary goes normally implies Clive goes.

- From (Ref) & (RW) Supraclassicality follows:

  \[
  \alpha \not\models \alpha + \models \alpha \to \beta, \alpha \not\models \alpha \\
  \alpha \not\models \beta \\
  \alpha \models \beta
  \]
Desirable properties: Right Weakening

**Right Weakening (RW):**

\[ \models \alpha \rightarrow \beta, \gamma \vdash \alpha \]
\[ \gamma \vdash \beta \]

- **Rationale:** If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example:** Assume that
  Mary goes normally implies Clive goes and John goes. Then we would expect that
  Mary goes normally implies Clive goes.

- From (Ref) & (RW) **Supraclassicality** follows:

  \[ \alpha \vdash \alpha + \models \alpha \rightarrow \beta, \alpha \vdash \alpha \]
  \[ \alpha \vdash \beta \]
Right Weakening in default logic

Proposition

Default logic satisfies Right Weakening.

Proof.

Assume \( \models \alpha \rightarrow \beta \) and \( \gamma \vdash_{\Delta} \alpha \) (with \( \Delta = \langle D, W \rangle \)).

Hence, \( \alpha \) is in each extension \( E \) of the default theory \( \langle D, W \cup \{ \gamma \} \rangle \).

Since extensions are closed under logical consequence, \( \beta \) must also be in each extension of \( \langle D, W \cup \{ \gamma \} \rangle \).

Hence, \( \gamma \vdash_{\Delta} \beta \).
Right Weakening in default logic

**Proposition**

*Default logic satisfies Right Weakening.*

**Proof.**

Assume $\models \alpha \rightarrow \beta$ and $\gamma \not\Delta \alpha$ (with $\Delta = \langle D, W \rangle$). Hence, $\alpha$ is in each extension $E$ of the default theory $\langle D, W \cup \{\gamma}\rangle$. Since extensions are closed under logical consequence, $\beta$ must also be in each extension of $\langle D, W \cup \{\gamma}\rangle$. Hence, $\gamma \not\Delta \beta$
Proposition

*Default logic satisfies Right Weakening.*

Proof.

Assume \( \models \alpha \rightarrow \beta \) and \( \gamma \vdash_{\Delta} \alpha \) (with \( \Delta = \langle D, W \rangle \)). Hence, \( \alpha \) is in each extension \( E \) of the default theory \( \langle D, W \cup \{ \gamma \} \rangle \). Since extensions are closed under logical consequence, \( \beta \) must also be in each extension of \( \langle D, W \cup \{ \gamma \} \rangle \).

Hence, \( \gamma \vdash_{\Delta} \beta \)
Proposition

Default logic satisfies Right Weakening.

Proof.

Assume $\models \alpha \rightarrow \beta$ and $\gamma \sim_{\Delta} \alpha$ (with $\Delta = \langle D, W \rangle$).

Hence, $\alpha$ is in each extension $E$ of the default theory $\langle D, W \cup \{\gamma\} \rangle$.

Since extensions are closed under logical consequence, $\beta$ must also be in each extension of $\langle D, W \cup \{\gamma\} \rangle$.

Hence, $\gamma \sim_{\Delta} \beta$
Desirable properties: Cut

\[
\frac{\alpha \vdash \beta, \alpha \land \beta \vdash \gamma}{\alpha \vdash \gamma}
\]

- **Rationale**: If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.

- **Example**: Assume that
  John goes normally implies Mary goes.
  Assume further that
  John goes and Mary goes normally implies Clive goes.
  Then we would expect that
  John goes normally implies Clive goes.
Proposition

**Default logic satisfies Cut.**

**Proof idea.**

Assume $\alpha \models_\Delta \beta$ (with $\Delta = \langle D, W \rangle$). Hence $\beta$ is contained in each extension of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$. Show that every extension $E$ of $\Delta'$ is also an extension of $\Delta'' = \langle D, W \cup \{\alpha \land \beta\} \rangle$.

- Consistency of justifications of defaults is tested against $E$ both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \land \beta\}$ case.
- The preconditions that are derivable when starting from $W \cup \{\alpha\}$ are also derivable when starting from $W \cup \{\alpha \land \beta\}$.
- $W \cup \{\alpha \land \beta\}$ does not allow for deriving further preconditions because also in the $W \cup \{\alpha\}$ case at some point $\beta$ is derived.

Hence, because $\gamma$ belongs to all extensions of $\Delta'' (\alpha \land \beta \models \gamma)$, it also belongs to all extensions of $\Delta' (\alpha \models \gamma)$.
Proposition

Default logic satisfies Cut.

Proof idea.

Assume $\alpha \models_{\Delta} \beta$ (with $\Delta = \langle D, W \rangle$). Hence $\beta$ is contained in each extension of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$. Show that every extension $E$ of $\Delta'$ is also an extension of $\Delta'' = \langle D, W \cup \{\alpha \land \beta\} \rangle$.

- Consistency of justifications of defaults is tested against $E$ both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \land \beta\}$ case.
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- $W \cup \{\alpha \land \beta\}$ does not allow for deriving further preconditions because also in the $W \cup \{\alpha\}$ case at some point $\beta$ is derived.

Hence, because $\gamma$ belongs to all extensions of $\Delta'' (\alpha \land \beta \models_{\Delta''} \gamma)$, it also belongs to all extensions of $\Delta' (\alpha \models_{\Delta'} \gamma)$. 
Proposition

Default logic satisfies Cut.

Proof idea.

Assume $\alpha \mid \sim_\Delta \beta$ (with $\Delta = \langle D, W \rangle$). Hence $\beta$ is contained in each extension of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$. Show that every extension $E$ of $\Delta'$ is also an extension of $\Delta'' = \langle D, W \cup \{\alpha \land \beta\} \rangle$.

- Consistency of justifications of defaults is tested against $E$ both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \land \beta\}$ case.
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- $W \cup \{\alpha \land \beta\}$ does not allow for deriving further preconditions because also in the $W \cup \{\alpha\}$ case at some point $\beta$ is derived.

Hence, because $\gamma$ belongs to all extensions of $\Delta''$ ($\alpha \land \beta \mid \sim \gamma$), it also belongs to all extensions of $\Delta'$ ($\alpha \mid \sim \gamma$).
Desirable properties:
Cautious Monotonicity

Cautious Monotonicity (CM):

\[ \alpha \not\vdash \beta, \alpha \not\vdash \gamma \]
\[ \frac{}{\alpha \land \beta \not\vdash \gamma} \]

- **Rationale**: In general, adding new premises may cancel some conclusions. However, existing conclusions may be added to the premises without canceling any conclusions!

- **Example**: Assume that
  
  Mary goes normally implies Clive goes and
  Mary goes normally implies John goes.
  Mary goes and Jack goes might not normally imply that John goes.
  However, Mary goes and Clive goes should normally imply that John goes.
Cautious Monotonicity in default logic

Proposition

Default logic does not satisfy Cautious Monotonicity.

Proof.

Consider the default theory \( \langle D, W \rangle \) with

\[
D = \left\{ \frac{a : g}{g}, \frac{g : b}{b}, \frac{b : \neg g}{\neg g} \right\}
\]

and \( W = \{a\} \).

\( E = \text{Th}(\{a, b, g\}) \) is the only extension of \( \langle D, W \rangle \) and thus both \( b \) and \( g \) follow skeptically (i.e., we have \( a \vdash_{\langle D, \emptyset \rangle} b \) and \( a \vdash_{\langle D, \emptyset \rangle} g \)).

For \( \langle D, \{a \land b\} \rangle \) also \( \text{Th}(\{a, b, \neg g\}) \) is an extension, and thus \( g \) does not follow skeptically (i.e., \( a \land b \not\vdash_{\langle D, \emptyset \rangle} g \)).
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Consider the default theory $\langle D, W \rangle$ with

\[ D = \left\{ \frac{a : g}{g}, \frac{g : b}{b}, \frac{b : \neg g}{\neg g} \right\} \quad \text{and} \quad W = \{a\}. \]

$E = \text{Th}(\{a, b, g\})$ is the only extension of $\langle D, W \rangle$ and thus both $b$ and $g$ follow skeptically (i.e., we have $a \vdash_{\langle D, \emptyset \rangle} b$ and $a \vdash_{\langle D, \emptyset \rangle} g$).

For $\langle D, \{a \land b\} \rangle$ also $\text{Th}(\{a, b, \neg g\})$ is an extension, and thus $g$ does not follow skeptically (i.e., $a \land b \not\vdash_{\langle D, \emptyset \rangle} g$).
Cumulativity

Lemma

Rules (Cut) & (CM) can be equivalently stated as follows:

If $\alpha \not\sim \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

This property is called Cumulativity.

Proof.

$\Rightarrow$: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \not\sim \beta$.
Assume further that $\alpha \not\sim \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\sim \gamma$. Similarly, by applying (Cut), from $\alpha \land \beta \not\sim \gamma$ it follows $\alpha \not\sim \gamma$.
Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$: Assume Cumulativity and $\alpha \not\sim \beta$. Now we can derive both rules (Cut) and (CM).
Lemma

Rules (Cut) & (CM) can be equivalently stated as follows:

If $\alpha \not\models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

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Assume further that $\alpha \not\models \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\models \gamma$. Similarly, by applying (Cut), from $\alpha \land \beta \not\models \gamma$ it follows $\alpha \not\models \gamma$.

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Assume further that $\alpha \not\models \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\models \gamma$. Similarly, by applying (Cut), from $\alpha \land \beta \not\models \gamma$ it follows $\alpha \not\models \gamma$.

Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

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Assume further that $\alpha \not \sim \gamma$. By applying (CM), we obtain $\alpha \land \beta \not \sim \gamma$.

Similarly, by applying (Cut), from $\alpha \land \beta \not \sim \gamma$ it follows $\alpha \not \sim \gamma$.

Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$: Assume **Cumulativity** and $\alpha \not \sim \beta$. Now we can derive both rules (Cut) and (CM).
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Proof.

$\Rightarrow$: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \not\sim \beta$.
Assume further that $\alpha \not\sim \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\sim \gamma$.
Similarly, by applying (Cut), from $\alpha \land \beta \not\sim \gamma$ it follows $\alpha \not\sim \gamma$.

Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$: Assume **Cumulativity** and $\alpha \not\sim \beta$. Now we can derive both rules (Cut) and (CM).
Lemma

Rules (Cut) & (CM) can be equivalently stated as follows:

If $\alpha \not\rightarrow \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

This property is called **Cumulativity**.

Proof.

$\Rightarrow$: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \not\rightarrow \beta$.
Assume further that $\alpha \not\rightarrow \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\rightarrow \gamma$. Similarly, by applying (Cut), from $\alpha \land \beta \not\rightarrow \gamma$ it follows $\alpha \not\rightarrow \gamma$.
Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$: Assume Cumulativity and $\alpha \not\rightarrow \beta$. Now we can derive both rules (Cut) and (CM).
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Rules (Cut) & (CM) can be equivalently stated as follows:

If $\alpha \not\models \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

This property is called **Cumulativity**.

Proof.

$\Rightarrow$: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \not\models \beta$. Assume further that $\alpha \not\models \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\models \gamma$. Similarly, by applying (Cut), from $\alpha \land \beta \not\models \gamma$ it follows $\alpha \not\models \gamma$. Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$: Assume **Cumulativity** and $\alpha \not\models \beta$. Now we can derive both rules (Cut) and (CM).
Lemma

Rules (Cut) & (CM) can be equivalently stated as follows:

If $\alpha \not\sim \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

This property is called Cumulativity.

Proof.

⇒: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \not\sim \beta$.
Assume further that $\alpha \not\sim \gamma$. By applying (CM), we obtain $\alpha \land \beta \not\sim \gamma$. Similarly, by applying (Cut), from $\alpha \land \beta \not\sim \gamma$ it follows $\alpha \not\sim \gamma$.
Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

⇐: Assume Cumulativity and $\alpha \not\sim \beta$. Now we can derive both rules (Cut) and (CM).
The System C

1. Reflexivity

\[
\alpha \not\sim \alpha
\]

2. Left Logical Equivalence

\[
\models \alpha \leftrightarrow \beta, \alpha \not\sim \gamma
\]

\[
\beta \not\sim \gamma
\]

3. Right Weakening

\[
\models \alpha \rightarrow \beta, \gamma \not\sim \alpha
\]

\[
\gamma \not\sim \beta
\]

4. Cut

\[
\alpha \not\sim \beta, \alpha \land \beta \not\sim \gamma
\]

\[
\alpha \not\sim \gamma
\]

5. Cautious Monotonicity

\[
\alpha \not\sim \beta, \alpha \not\sim \gamma
\]

\[
\alpha \land \beta \not\sim \gamma
\]
Derived rules in C

- **Equivalence:**
  \[ \alpha \sim \beta, \beta \sim \alpha, \alpha \sim \gamma \]
  \[ \Rightarrow \quad \beta \sim \gamma \]

- **And:**
  \[ \alpha \sim \beta, \alpha \sim \gamma \]
  \[ \Rightarrow \quad \alpha \sim \beta \land \gamma \]

- **MPC:**
  \[ \alpha \sim \beta \rightarrow \gamma, \alpha \sim \beta \]
  \[ \Rightarrow \quad \alpha \sim \gamma \]
Derived rules: proofs

Proof (Equivalence).

Assumption: \( \alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \vdash \gamma \)

Left L Equivalence: \( \beta \land \alpha \vdash \gamma \)

\[ \begin{align*}
\text{Cut:} & \quad \beta \vdash \gamma
\end{align*} \]

Proof (And).

Assumption: \( \alpha \vdash \beta, \alpha \vdash \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \vdash \gamma \)

Left L Equivalence: \( \beta \land \alpha \vdash \gamma \)

\[ \begin{align*}
\text{Cut:} & \quad \beta \vdash \gamma
\end{align*} \]

MPC is an exercise.
### Derived rules: proofs

#### Proof (Equivalence).

**Assumption:** \( \alpha \not\models \beta, \ \beta \not\models \alpha, \ \alpha \not\models \gamma \)

**Cautious Monotonicity:** \( \alpha \wedge \beta \not\models \gamma \)

**Left L Equivalence:** \( \beta \wedge \alpha \not\models \gamma \)

**Cut:** \( \beta \not\models \gamma \)

#### Proof (And).

**Assumption:** \( \alpha \not\models \beta, \ \alpha \not\models \gamma \)

**Cautious Monotonicity:** \( \alpha \wedge \beta \not\models \gamma \)

**propositional logic:** \( \alpha \wedge \beta \wedge \gamma \models \beta \wedge \gamma \)

**Supraclasiicality:** \( \alpha \wedge \beta \wedge \gamma \not\models \beta \wedge \gamma \)

**Cut:** \( \alpha \wedge \beta \not\models \beta \wedge \gamma \)

**Cut:** \( \alpha \not\models \beta \wedge \gamma \)

MPC is an exercise.
Derived rules: proofs

Proof (Equivalence).

Assumption: \( \alpha \not\models \beta, \ \beta \not\models \alpha, \ \alpha \not\models \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \not\models \gamma \)

Left L Equivalence: \( \beta \land \alpha \not\models \gamma \)

Cut: \( \beta \not\models \gamma \)

Proof (And).

Assumption: \( \alpha \not\models \beta, \ \alpha \not\models \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \not\models \gamma \)

propositional logic: \( \alpha \land \beta \land \gamma \models \beta \land \gamma \)

Supraclassicality: \( \alpha \land \beta \land \gamma \not\models \beta \land \gamma \)

Cut: \( \alpha \land \beta \not\models \beta \land \gamma \)

Cut: \( \alpha \not\models \beta \land \gamma \)

MPC is an exercise.
Derived rules: proofs

Proof (Equivalence).

Assumption: $\alpha \vdash \beta, \beta \vdash \alpha, \alpha \vdash \gamma$

Cautious Monotonicity: $\alpha \land \beta \vdash \gamma$

Left L Equivalence: $\beta \land \alpha \vdash \gamma$

Cut: $\beta \vdash \gamma$

Proof (And).

Assumption: $\alpha \vdash \beta, \alpha \vdash \gamma$

Cautious Monotonicity: $\alpha \land \beta \vdash \gamma$

Propositional logic: $\alpha \land \beta \land \gamma \models \beta \land \gamma$

Supraclassicality: $\alpha \land \beta \land \gamma \vdash \beta \land \gamma$

Cut: $\alpha \land \beta \vdash \beta \land \gamma$

Cut: $\alpha \vdash \beta \land \gamma$

MPC is an exercise.
Derived rules: proofs

Proof (Equivalence).

Assumption: \( \alpha \models \sim \beta, \ \beta \models \sim \alpha, \ \alpha \models \sim \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \models \sim \gamma \)

Left L Equivalence: \( \beta \land \alpha \models \sim \gamma \)

Cut: \( \beta \models \sim \gamma \)

Proof (And).

Assumption: \( \alpha \models \sim \beta, \ \alpha \models \sim \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \models \sim \gamma \)

propositional logic: \( \alpha \land \beta \land \gamma = \beta \land \gamma \)

Supraclasciallity: \( \alpha \land \beta \land \gamma \models \sim \beta \land \gamma \)

Cut: \( \alpha \land \beta \models \gamma \)

Cut: \( \alpha \models \sim \beta \land \gamma \)

MPC is an exercise.
Derived rules: proofs

Proof (Equivalence).

Assumption: \( \alpha \not\models \beta, \quad \beta \not\models \alpha, \quad \alpha \not\models \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \not\models \gamma \)

Left L Equivalence: \( \beta \land \alpha \not\models \gamma \)

Cut: \( \beta \not\models \gamma \)

Proof (And).

Assumption: \( \alpha \not\models \beta, \quad \alpha \not\models \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \not\models \gamma \)

propositional logic: \( \alpha \land \beta \land \gamma \models \beta \land \gamma \)

Supraclassicality: \( \alpha \land \beta \land \gamma \not\models \beta \land \gamma \)

Cut: \( \alpha \land \beta \not\models \beta \land \gamma \)

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MPC is an exercise.
Derived rules: proofs

Proof (Equivalence).

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MPC is an exercise.
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**MPC** is an exercise.
### Undesirable properties: Monotonicity and Contraposition

**Monotonicity:**

\[
\models \alpha \rightarrow \beta, \beta \not\models \gamma \\
\models \alpha \not\models \gamma
\]

**Example:** Let us assume that

John goes normally implies Mary goes.

Now we will probably not expect that

John goes and Joan (who is not in talking terms with Mary) goes normally implies Mary goes.

**Contraposition:**

\[
\alpha \not\models \beta \\
\models \neg \beta \not\models \neg \alpha
\]

**Example:** Let us assume that

John goes normally implies Mary goes.

Would we expect that

Mary does not go normally implies John does not go?

What if John goes always?
Undesirable properties: Monotonicity

\[ \alpha \models \beta, \beta \not\sim \gamma, \text{ but not } \alpha \not\sim \gamma \] — pictorially:

![Diagram showing the relationship between \( \alpha \), \( \beta \), and \( \gamma \).]
Undesirable properties: Contraposition

$$\alpha \not\sim \beta$$, but not $$\neg \beta \not\sim \neg \alpha$$ — pictorially:
Undesirable properties: Transitivity & EHD

- **Transitivity:**
  \[ \alpha \not\implies \beta, \beta \not\implies \gamma \]
  \[ \therefore \alpha \not\implies \gamma \]

- **Example:** Let us assume that
  - John goes *normally* implies Mary goes and
  - Mary goes *normally* implies Jack goes.
  Now, should John goes *normally imply* that Jack goes? What, if John goes very seldom?

- **Easy Half of the Deduction Theorem (EHD):**
  \[ \alpha \not\implies \beta \rightarrow \gamma \]
  \[ \therefore \alpha \land \beta \not\implies \gamma \]
Undesirable properties: Transitivity

\( \alpha \not\sim \beta, \beta \sim \gamma, \text{ but not } \alpha \sim \gamma \) — pictorially:
Undesirable properties: EHD

$\alpha \not\models \beta \rightarrow \gamma$, but not $\alpha \land \beta \not\models \gamma$ — pictorially:
Monotonicity vs EHD

**Theorem**

*In the presence of the rules in system C, the rules Monotonicity and EHD are equivalent.*

**Proof.**

**Monotonicity $\Rightarrow$ EHD:**

- $\alpha \not\models \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\models \beta \rightarrow \gamma$ (Monotonicity)
- $\alpha \land \beta \not\models \alpha \land \beta$ (Ref)
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December 7 & 12 2012 Nebel, Wölf, Hué – KRR
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- $\alpha \land \beta \nvdash \gamma$ (Monotonicity)
- $\alpha \nvdash \gamma$ (Cut)

Monotonicity $\Leftarrow$ Transitivity:
- $\models \alpha \rightarrow \beta, \beta \nvdash \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \nvdash \beta$ (Supraclassicality)
- $\alpha \nvdash \gamma$ (Transitivity)
Theorem

In the presence of the rules in system \( C \), the rules Monotonicity and Transitivity are equivalent.

Proof.

Monotonicity \( \Rightarrow \) Transitivity:

\[ \alpha \not\models \beta, \beta \not\models \gamma \text{ (assumption)} \]
\[ \alpha \land \beta \not\models \gamma \text{ (Monotonicity)} \]
\[ \alpha \not\models \gamma \text{ (Cut)} \]

Monotonicity \( \Leftarrow \) Transitivity:

\[ \models \alpha \rightarrow \beta, \beta \not\models \gamma \text{ (assumption)} \]
\[ \alpha \models \beta \text{ (deduction theorem)} \]
\[ \alpha \not\models \beta \text{ (Supraclassicality)} \]
\[ \alpha \not\models \gamma \text{ (Transitivity)} \]
Theorem

In the presence of Right Weakening, Contraposition implies Monotonicity.

Proof.

\[ \alpha \rightarrow \beta, \beta \not\rightarrow \gamma \text{ (assumption)} \]

\[ \neg \gamma \not\rightarrow \neg \beta \text{ (Contraposition)} \]

\[ \alpha \not\rightarrow \neg \gamma \text{ (classical contraposition)} \]

\[ \neg \gamma \not\rightarrow \neg \alpha \text{ (RW)} \]

\[ \alpha \not\rightarrow \gamma \text{ (Contraposition)} \]

Note: Monotonicity does not imply Contraposition, even in the presence of all rules of system C!
**Theorem**

*In the presence of Right Weakening, Contraposition implies Monotonicity.*

**Proof.**

- $\vdash \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\neg \gamma \not\models \neg \beta$ (Contraposition)
- $\vdash \neg \beta \rightarrow \neg \alpha$ (classical contraposition)
- $\neg \gamma \not\models \neg \alpha$ (RW)
- $\alpha \not\models \gamma$ (Contraposition)

Note: Monotonicity does not imply Contraposition, even in the presence of all rules of system $C$!
**Theorem**

*In the presence of Right Weakening, Contraposition implies Monotonicity.*

**Proof.**

- $\alpha \implies \beta, \beta \not\vdash \gamma$ (assumption)
- $\neg \gamma \not\vdash \neg \beta$ (Contraposition)
- $\not\vdash \neg \beta \implies \neg \alpha$ (classical contraposition)
- $\neg \gamma \not\vdash \neg \alpha$ (RW)
- $\alpha \not\vdash \gamma$ (Contraposition)

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### Theorem

*In the presence of Right Weakening, Contraposition implies Monotonicity.*

### Proof.

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\neg \gamma \models \neg \beta$ (Contraposition)
- $\models \neg \beta \rightarrow \neg \alpha$ (classical contraposition)
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**Theorem**

In the presence of Right Weakening, Contraposition implies **Monotonicity**.

**Proof.**

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Contraposition?

Theorem

*In the presence of Right Weakening, Contraposition implies Monotonicity.*

Proof.

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\not\models \gamma \not\models \beta$ (Contraposition)
- $\models \not\models \beta \rightarrow \not\models \alpha$ (classical contraposition)
- $\not\models \gamma \not\models \not\models \alpha$ (RW)
- $\models \alpha \not\models \gamma$ (Contraposition)

Note: Monotonicity does not imply Contraposition, even in the presence of all rules of system $\mathbf{C}$!
Reasoning
How do we reason with $\models$ from $\varphi$ to $\psi$?

**Assumption:** We have some (finite) set $K$ of **conditional statements** of the form $\alpha \models \beta$.

The question is: Assuming the statements in $K$, is it plausible to conclude $\psi$ given $\varphi$?

**Idea:** We consider all cumulative consequence relations that contain $K$.

**Cumulative consequence relation:** any relation $\models$ between propositional logic formulae that is closed under the rules of system $\mathbf{C}$.

**Remark:** It suffices to consider only the **minimal** cumulative consequence relation containing $K$ . . .
Lemma

The set of cumulative consequence relations is closed under (arbitrary) intersections.

Proof.

Let $\sim_1$ and $\sim_2$ be cumulative consequence relations. We have to show that $\sim_1 \cap \sim_2$ is a cumulative consequence relation, that is, it is closed under all the rules of system $\mathbb{C}$.

Take any instance of any of the rules. If the preconditions are satisfied by $\sim_1$ and $\sim_2$, then the consequence is trivially also satisfied by both. A similar argument works if we consider an arbitrary family of consequence relations.
Lemma

The set of cumulative consequence relations is closed under (arbitrary) intersections.

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**Lemma**

The set of cumulative consequence relations is closed under (arbitrary) intersections.

**Proof.**

Let \( \sim_1 \) and \( \sim_2 \) be cumulative consequence relations. We have to show that \( \sim_1 \cap \sim_2 \) is a cumulative consequence relation, that is, it is closed under all the rules of system \( \mathbf{C} \).

Take any instance of any of the rules. If the preconditions are satisfied by \( \sim_1 \) and \( \sim_2 \), then the consequence is trivially also satisfied by both.

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The set of cumulative consequence relations is closed under (arbitrary) intersections.

Proof.

Let $\sim_1$ and $\sim_2$ be cumulative consequence relations. We have to show that $\sim_1 \cap \sim_2$ is a cumulative consequence relation, that is, it is closed under all the rules of system $C$.

Take any instance of any of the rules. If the preconditions are satisfied by $\sim_1$ and $\sim_2$, then the consequence is trivially also satisfied by both. A similar argument works if we consider an arbitrary family of consequence relations.
Theorem

*For each set of conditional statements $K$, there exists a unique minimal cumulative consequence relation containing $K$.*

Proof.

From the previous lemma it is clear that the intersection of all the cumulative consequence relations containing $K$ is already such a cumulative consequence relation. Obviously, there cannot be two distinct such minimal relations.

This relation is called the *cumulative closure* $K^c$ of $K$. 
Theorem

For each set of conditional statements $K$, there exists a unique minimal cumulative consequence relation containing $K$.

Proof.

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This relation is called the cumulative closure $K^C$ of $K$. 
Semantics
We will now try to characterize cumulative reasoning model-theoretically.

**Idea**: Cumulative models consist of states ordered by a preference relation.

**States** characterize beliefs.

The preference relation, \(\prec\), expresses the normality of the beliefs.

We read \(s \prec t\) as: state \(s\) is preferred to/more normal than state \(t\).

We say: \(\alpha \models \beta\) is accepted in a model if in all most preferred states in which \(\alpha\) is true also \(\beta\) is true.
We consider an arbitrary binary relation $≺$ on a given set of states $S$.
Later, we will assume that $≺$ has particular properties, e.g., that $≺$ is irreflexive, asymmetric, transitive, a partial order, ... 
... but currently we make no such restrictions.

We need a condition on state sets claiming that each state is, or is related to, a most preferred state.

**Definition (Smoothness)**

Let $P \subseteq S$.

- We say that $s \in P$ is **minimal in** $P$ if $s' \not≺ s$ for each $s' \in P$.
- $P$ is called **smooth** if for each $s \in P$, either $s$ is minimal in $P$ or there exists an $s'$ such that $s'$ is minimal in $P$ and $s' ≺ s$. 
We consider an arbitrary binary relation $\prec$ on a given set of states $S$.
Later, we will assume that $\prec$ has particular properties, e.g., that $\prec$ is irreflexive, asymmetric, transitive, a partial order, ... ... but currently we make no such restrictions.

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**Definition (Smoothness)**

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Cumulative Models – formally

Let $\mathcal{U}$ be the set of all possible worlds (i.e., propositional interpretations).

- A **cumulative model** is a triple $W = \langle S, l, \prec \rangle$ such that
  1. $S$ is a set of states,
  2. $l$ is a mapping $l : S \to 2^\mathcal{U}$, and
  3. $\prec$ is an arbitrary binary relation on $S$

  such that the **smoothness condition** is satisfied (see below).

- A state $s \in S$ satisfies a formula $\alpha$ ($s \models \alpha$) if $m \models \alpha$ for each propositional interpretation $m \in l(s)$.
  The set of states satisfying $\alpha$ is denoted by $\hat{\alpha}$.

- **Smoothness condition**: A cumulative model satisfies this condition if for all formulae $\alpha$, $\hat{\alpha}$ is smooth.
Consequence relation induced by a cumulative model

A cumulative model $W$ induces a consequence relation $\models_W$ as follows:

$$\alpha \models_W \beta \text{ iff } s \equiv \beta \text{ for every minimal } s \text{ in } \hat{\alpha}.$$ 

Example

Model $W = \langle \{s_1, s_2, s_3\}, l, \prec \rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

- $l(s_1) = \{\neg p, b, f\}$
- $l(s_2) = \{p, b, \neg f\}$
- $l(s_3) = \{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}$

Does $W$ satisfy the smoothness condition?

- $\neg p \land \neg b \models f$? N
- Also: $\neg p \land \neg b \not\models \neg f$!
- $p \models \neg f$? Y
- $\neg p \models f$? Y
Consequence relation induced by a cumulative model

A cumulative model $W$ induces a consequence relation $\sim_W$ as follows:

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Does $W$ satisfy the smoothness condition?

- $\neg p \land \neg b \not\sim f$? N
- $p \not\sim \neg f$? Y
- $\neg p \not\sim f$? Y

Also: $\neg p \land \neg b \not\prec \neg f$!
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Consequence relation induced by a cumulative model

A cumulative model $\mathcal{W}$ induces a consequence relation $\models_{\mathcal{W}}$ as follows:

$$\alpha \models_{\mathcal{W}} \beta \text{ iff } s \models \beta \text{ for every minimal } s \text{ in } \hat{\alpha}.$$ 

**Example**

Model $\mathcal{W} = \langle \{s_1, s_2, s_3\}, l, \prec \rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

\[\begin{align*}
    l(s_1) &= \{\neg p, b, f\} \\
    l(s_2) &= \{p, b, \neg f\} \\
    l(s_3) &= \{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}
\end{align*}\]

Does $\mathcal{W}$ satisfy the smoothness condition?

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Example

Model $W = \langle \{s_1, s_2, s_3\}, l, \prec \rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

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Does $W$ satisfy the smoothness condition?

- $\neg p \land \neg b \not\sim f$? \quad N  \quad Also: $\neg p \land \neg b \not|\not f$!
- $p \not\sim \neg f$? \quad Y
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Consequence relation induced by a cumulative model

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$p \sim \neg f$? Y

$\neg p \sim f$? Y
Soundness 1

Theorem

If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- Reflexivity: satisfied.
- LLE: satisfied.
- RW: satisfied.
- Cut: $\alpha \sim_W \beta$, $\alpha \land \beta \sim_W \gamma \Rightarrow \alpha \sim_W \gamma$. Assume that all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\alpha \land \beta$ satisfy $\gamma$. Every minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\alpha \land \beta \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\alpha \land \beta$. Hence $\alpha \sim_W \gamma$. 

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Theorem

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Soundness 1

Theorem

If $W$ is a cumulative model, then $\not\sim_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied.
- **LLE:** satisfied.
- **RW:** satisfied.
- **Cut:** $\alpha \not\sim_W \beta$, $\alpha \land \beta \not\sim_W \gamma \Rightarrow \alpha \not\sim_W \gamma$. Assume that all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\alpha \land \beta$ satisfy $\gamma$. Every minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\alpha \land \beta \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\alpha \land \beta$. Hence $\alpha \not\sim_W \gamma$. 
Theorem

If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied.
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- **Cut:** $\alpha \sim_W \beta, \alpha \land \beta \sim_W \gamma \Rightarrow \alpha \sim_W \gamma$. Assume that all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\alpha \land \beta$ satisfy $\gamma$. Every minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\alpha \land \beta \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\alpha \land \beta$. Hence $\alpha \sim_W \gamma$. 
Soundness 1

Theorem

If $W$ is a cumulative model, then $\not\models_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied.
- **LLE:** satisfied.
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Theorem

If $W$ is a cumulative model, then $\not\in_W$ is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied.
- **LLE:** satisfied.
- **RW:** satisfied.
- **Cut:** $\alpha \not\in_W \beta, \alpha \land \beta \not\in_W \gamma \Rightarrow \alpha \not\in_W \gamma$. Assume that all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\alpha \land \beta$ satisfy $\gamma$. Every minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\alpha \land \beta \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\alpha \land \beta$. Hence $\alpha \not\in_W \gamma$. 

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Theorem

If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

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Theorem

If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

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Soundness 1

Theorem

*If W is a cumulative model, then \( \sim_W \) is a cumulative consequence relation.*

Proof.

- **Reflexivity:** satisfied.
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- **Cut:** \( \alpha \sim_W \beta, \alpha \land \beta \sim_W \gamma \Rightarrow \alpha \sim_W \gamma \). Assume that all minimal elements of \( \hat{\alpha} \) satisfy \( \beta \), and all minimal elements of \( \hat{\alpha} \land \hat{\beta} \) satisfy \( \gamma \). Every minimal element of \( \hat{\alpha} \) satisfies \( \alpha \land \beta \). Since \( \alpha \land \beta \subseteq \hat{\alpha} \), all minimal elements of \( \hat{\alpha} \) are also minimal elements of \( \hat{\alpha} \land \hat{\beta} \). Hence \( \alpha \sim_W \gamma \).
Theorem

If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

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- **Reflexivity**: satisfied.
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Soundness 1

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If $W$ is a cumulative model, then $\sim_W$ is a cumulative consequence relation.

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- **Reflexivity**: satisfied.
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- **Cut**: $\alpha \sim_W \beta$, $\alpha \land \beta \sim_W \gamma \Rightarrow \alpha \sim_W \gamma$. Assume that all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. Every minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\hat{\alpha} \land \hat{\beta} \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\alpha \land \beta$. Hence $\alpha \sim_W \gamma$. 
Cautious Monotonicity: \((\alpha \not\models \beta, \alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma)\)

Assume \(\alpha \not\models_W \beta\) and \(\alpha \not\models_W \gamma\). We have to show: \(\alpha \land \beta \not\models_W \gamma\), i.e., \(s \equiv \gamma\) for all minimal \(s \in \alpha \land \beta\).

Clearly, every minimal \(s \in \alpha \land \beta\) is in \(\hat{\alpha}\).

We show that every minimal \(s \in \alpha \land \beta\) is minimal in \(\hat{\alpha}\).

Assumption: There is \(s\) that is minimal in \(\alpha \land \beta\), but not minimal in \(\hat{\alpha}\). Because of smoothness there is minimal \(s' \in \hat{\alpha}\) such that \(s' \prec s\). We know, however, that \(s' \models \beta\), which means that \(s' \in \alpha \land \beta\). Hence \(s\) is not minimal in \(\alpha \land \beta\). Contradiction!

Hence \(s\) must be minimal in \(\hat{\alpha}\), and therefore \(s \equiv \gamma\). Because this is true for all minimal elements in \(\alpha \land \beta\), we get \(\alpha \land \beta \not\models_W \gamma\). □
Soundness 2

Proof continues...

- **Cautious Monotonicity**: $(\alpha \not\models \beta, \alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma)$

  Assume $\alpha \not\models_W \beta$ and $\alpha \not\models_W \gamma$. We have to show: $\alpha \land \beta \not\models_W \gamma$, i.e., $s \models \gamma$ for all minimal $s \in \hat{\alpha} \land \hat{\beta}$.

  Clearly, every minimal $s \in \hat{\alpha} \land \hat{\beta}$ is in $\hat{\alpha}$.

  We show that every minimal $s \in \hat{\alpha} \land \hat{\beta}$ is minimal in $\hat{\alpha}$.

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Cautious Monotonicity: \((\alpha \not\sim \beta, \alpha \not\sim \gamma \Rightarrow \alpha \land \beta \not\sim \gamma)\)

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Soundness 2

Proof continues...

Cautious Monotonicity: \((\alpha \not\models \beta, \alpha \not\models \gamma \Rightarrow \alpha \land \beta \not\models \gamma)\)

Assume \(\alpha \not\models_W \beta\) and \(\alpha \not\models_W \gamma\). We have to show: \(\alpha \land \beta \not\models_W \gamma\), i.e., \(s \models \gamma\) for all minimal \(s \in \alpha \land \beta\).

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Proof continues...

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Hence $s$ must be minimal in $\hat{\alpha}$, and therefore $s \models \gamma$. Because this is true for all minimal elements in $\alpha \land \beta$, we get $\alpha \land \beta \not\sim_W \gamma$. □
Now we have a method for showing that a principle does not hold for cumulative consequence relations: 
... construct a cumulative model that falsifies the principle.

**Contraposition**: $\alpha \not\vdash \beta \Rightarrow \neg\beta \not\vdash \neg\alpha$

$W = \langle S, I, \prec \rangle$

$S = \{s_1, s_2\}$

$s_i \not\prec s_j \ \forall s_i, s_j \in S$

$I(s_1) = \{\{a, b\}\}$

$I(s_2) = \{\{a, \neg b\}, \{\neg a, \neg b\}\}$

$W$ is a cumulative model with $a \vdash_{W} b$, but $\neg b \not\vdash_{W} \neg a$. 
Completeness?

- Each cumulative model \( W \) induces a cumulative consequence relation \( \sim_W \).
- **Problem**: Can we generate all cumulative consequence relations in this way?
- We can! There is a representation theorem:

**Theorem (Representation of cumulative consequence)**

A consequence relation is cumulative if and only if it is induced by some cumulative model.

\[ \sim \] Cumulative consequence can be characterized independently from the set of inference rules.
Could we strengthen the preference relation to transitive relations without sacrificing anything? 

No!

In such models, the following additional principle called Loop is valid:

\[
\alpha_0 \sim \alpha_1, \alpha_1 \sim \alpha_2, \ldots, \alpha_k \sim \alpha_0 \\
\alpha_0 \sim \alpha_k
\]

For the system \( \textbf{CL} = \textbf{C} + \text{(Loop)} \) and cumulative models with transitive preference relations, we could prove another representation theorem.
The Or Rule

**Or rule:**

\[
\frac{\alpha \not\models \gamma, \beta \not\models \gamma}{\alpha \lor \beta \not\models \gamma}
\]

Not valid in system C. **Counterexample:**

\[\mathcal{W} = \langle S, l, \prec \rangle\]

\[S = \{s_1, s_2, s_3\}, s_i \not\prec s_j \forall s_i, s_j \in S\]

\[l(s_1) = \{\{a, b, c\}, \{a, \neg b, c\}\}\]

\[l(s_2) = \{\{a, b, c\}, \{\neg a, b, c\}\}\]

\[l(s_3) = \{\{a, b, \neg c\}, \{a, \neg b, \neg c\}, \{\neg a, b, \neg c\}\}\]

\[a \not\models_{\mathcal{W}} c, b \not\models_{\mathcal{W}} c, \text{ but not } a \lor b \not\models_{\mathcal{W}} c.\]

**Note:** Or is not valid in default logic.
Preferential Reasoning
System $\mathbf{P}$

- System $\mathbf{P}$ contains all rules of $\mathbf{C}$ and the $\text{Or}$ rule.
- A consequence relation that satisfies $\mathbf{P}$ is called preferential.
- Derived rules in $\mathbf{P}$:
  - Hard half of the deduction theorem ($S$):
    \[
    \frac{\alpha \land \beta \models \gamma}{\frac{\alpha \models \beta \rightarrow \gamma}{\alpha \models \gamma}}
    \]
  - Proof by case analysis ($D$):
    \[
    \frac{\alpha \land \neg \beta \models \gamma, \alpha \land \beta \models \gamma}{\frac{\alpha \models \gamma}{}}
    \]
- $D$ and $\text{Or}$ are equivalent in the presence of the rules in $\mathbf{C}$. 
A cumulative model $W = \langle S, I, \prec \rangle$ such that $\prec$ is a strict partial order (irreflexive and transitive) and $|I(s)| = 1$ for all $s \in S$ is called a preferential model.
Theorem (Soundness)

The consequence relation $\models_W$ induced by a preferential model is preferential.

Proof.

Since $W$ is cumulative, we only have to verify that Or holds. Note that in preferential models we have $\alpha \lor \beta = \hat{\alpha} \cup \hat{\beta}$. Suppose $\alpha \models_W \gamma$ and $\beta \models_W \gamma$. Because of the above equation, each minimal state of $\alpha \lor \beta$ is minimal in $\hat{\alpha} \cup \hat{\beta}$. Since $\gamma$ is satisfied in all minimal states in $\hat{\alpha} \cup \hat{\beta}$, $\gamma$ is also satisfied in all minimal states of $\alpha \lor \beta$. Hence $\alpha \lor \beta \models_W \gamma$. $\square$
Preferential models

**Theorem (Soundness)**

The consequence relation $\sim_w$ induced by a preferential model is preferential.

**Proof.**

Since $W$ is cumulative, we only have to verify that $\text{Or}$ holds. Note that in preferential models we have $\hat{\alpha} \cup \hat{\beta} = \hat{\alpha} \cup \hat{\beta}$. Suppose $\alpha \sim_w \gamma$ and $\beta \sim_w \gamma$. Because of the above equation, each minimal state of $\alpha \lor \beta$ is minimal in $\hat{\alpha} \cup \hat{\beta}$. Since $\gamma$ is satisfied in all minimal states in $\hat{\alpha} \cup \hat{\beta}$, $\gamma$ is also satisfied in all minimal states of $\alpha \lor \beta$. Hence $\alpha \lor \beta \sim_w \gamma$. \qed
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Proof.

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Theorem (Soundness)

The consequence relation \( \sim_W \) induced by a preferential model is preferential.

Proof.

Since \( W \) is cumulative, we only have to verify that Or holds. Note that in preferential models we have \( \alpha \lor \beta = \hat{\alpha} \cup \hat{\beta} \). Suppose \( \alpha \sim_W \gamma \) and \( \beta \sim_W \gamma \). Because of the above equation, each minimal state of \( \alpha \lor \beta \) is minimal in \( \hat{\alpha} \cup \hat{\beta} \). Since \( \gamma \) is satisfied in all minimal states in \( \hat{\alpha} \cup \hat{\beta} \), \( \gamma \) is also satisfied in all minimal states of \( \alpha \lor \beta \). Hence \( \alpha \lor \beta \sim_W \gamma \).
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*The consequence relation* $\models_W$ *induced by a preferential model is preferential.*

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The consequence relation $\Vdash_W$ induced by a preferential model is preferential.

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Theorem (Representation of preferential consequence)

A consequence relation is preferential if and only if it is induced by a preferential model.

Proof.

Similar to the one for $\mathcal{C}$.
### Summary of cumulative systems

<table>
<thead>
<tr>
<th>System</th>
<th>Models</th>
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<th><strong>CL</strong></th>
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<th><strong>P</strong></th>
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<td>+ Or</td>
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System C and System P do not produce many of the inferences one would hope for:

\[ \text{Given } K = \{ \text{Bird } \not\rightarrow \text{Flies} \} \text{ one cannot conclude } \text{Red} \land \text{Bird } \not\rightarrow \text{Flies}! \]

In general, adding information that is irrelevant cancels the plausible conclusions.

\[ \implies \text{Cumulative and Preferential consequence relations are too nonmonotonic.} \]

The plausible conclusions have to be strengthened!
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System C and System P do not produce many of the inferences one would hope for:

Given $K = \{ \text{Bird} \mid \neg \text{Flies} \}$ one cannot conclude $\text{Red} \land \text{Bird} \mid \neg \text{Flies}!$

In general, adding information that is irrelevant cancels the plausible conclusions.

$\implies$ Cumulative and Preferential consequence relations are too nonmonotonic.

The plausible conclusions have to be strengthened!
Strengthening the consequence relation

- System $C$ and System $P$ do not produce many of the inferences one would hope for:
  
  Given $K = \{Bird \sim Flies\}$ one cannot conclude $Red \land Bird \sim Flies$!

- In general, adding information that is irrelevant cancels the plausible conclusions.
  
  $\implies$ Cumulative and Preferential consequence relations are too nonmonotonic.

- The plausible conclusions have to be strengthened!
Strengthening the consequence relations

- The rules so far seem to be reasonable: one cannot think of rules of the same form (if we have some plausible implications, other plausible implications should hold) that could be added.

- However, there are other types of rules one might want to add.

- **Disjunctive Rationality:**

  \[
  \alpha \not\sim \gamma, \beta \not\sim \gamma \\
  \frac{}{\alpha \lor \beta \not\sim \gamma}
  \]

- **Rational Monotonicity:**

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  \alpha \not\sim \gamma, \alpha \not\sim \neg \beta \\
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- **Note:** Consequence relations obeying these rules are not closed under intersection, which is a problem.
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  \[ \alpha \vee \beta \not\models \gamma \]

- **Rational Monotonicity:**
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  \[ \alpha \wedge \beta \not\models \gamma \]

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- **Note:** Consequence relations obeying these rules are not closed under intersection, which is a problem.
Instead of ad hoc extensions of the logical machinery, analyze the properties of nonmonotonic consequence relations.

Correspondence between rule system and models for System $\mathbf{C}$, and for System $\mathbf{P}$ could also be established wrt. a probabilistic semantics.

Irrelevant information poses a problem. Solution approaches: rational monotonicity, maximum entropy approach.
Literature
Introduces cumulative consequence relations.

Introduces rational consequence relations.

First to consider abstract properties of nonmonotonic consequence relations.
Literature II

Judea Pearl.
Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference,
One section on $\varepsilon$-semantics and maximum entropy.

Yoav Shoham.
Reasoning about Change.
Introduces the idea of preferential models.