Principles of Knowledge Representation and Reasoning

Nonmonotonic Reasoning III: Cumulative Logics

Albert-Ludwigs-Universität Freiburg

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Introduction

Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models).
- Nonmonotonicity is only a negative characterization: From $\Theta \mid\!\sim \varphi$, it does not necessarily follow $\Theta \cup \{\psi\} \mid\!\sim \varphi$.
- Could we have a constructive positive characterization of default reasoning?

Plausible consequences

- In classical logics, we have the logical consequence relation $\alpha \models \beta$: If α is true, then also β is true.
- Instead, we will study the relation of plausible consequence $\alpha \mid\sim \beta$: If α is all we know, can we conclude β?
- $\alpha \mid\sim \beta$ does not imply $\alpha \land \alpha' \mid\sim \beta$!
  Compare to conditional probability: $P(\beta|\alpha) \neq P(\beta|\alpha,\alpha')$!
- Find rules that characterize $\mid\sim$ …
  For example: if $\alpha \mid\sim \beta$ and $\alpha \mid\sim \gamma$, then $\alpha \mid\sim \beta \land \gamma$.
- Write down all such rules …
- … and find a semantic characterization of $\mid\sim$!

Desirable properties: Reflexivity

Reflexivity (Ref):

$$\frac{}{\alpha \mid\!\sim \alpha}$$

- *Rationale*: If α holds, this normally implies α.
- *Example*: `Tom goes to a party` normally implies that `Tom goes to a party.`

Reflexivity in default logic

Let $\Delta = \langle D, W \rangle$ be a propositional default theory.
Define the relation $\mid\!\sim_\Delta$ as follows:

$$\alpha \mid\!\sim_\Delta \beta \iff \langle D, W \cup \{\alpha\} \rangle \mid\!\sim \beta$$

$\alpha \mid\!\sim_\Delta \beta$ means that β is a skeptical conclusion of $\langle D, W \cup \{\alpha\} \rangle$.

Proposition

Default logic satisfies Reflexivity.

Proof.

The question is: does α follow skeptically from $\Delta' = \langle D, W \cup \{\alpha\} \rangle$?
For each extension E of Δ', it holds $W \cup \{\alpha\} \subseteq E$ (by definition).
Hence $\alpha \in E$, and thus α belongs to all extensions of Δ'. $\square$

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Desirable properties: Left Logical Equivalence

Left Logical Equivalence (LLE):

$$\frac{\models \alpha \leftrightarrow \beta,\ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma}$$

- *Rationale*: It is not the syntactic form, but the content that is responsible for what we conclude normally.
- *Example*: Assume that
  `Tom goes` or `Peter goes` normally implies `Mary goes`.
  Then we would expect that
  `Peter goes` or `Tom goes` normally implies `Mary goes`.

Left Logical Equivalence in default logic

Proposition

Default logic satisfies Left Logical Equivalence.

Proof.

Assume $\models \alpha \leftrightarrow \beta$ and $\alpha \mid\!\sim_\Delta \gamma$ (with $\Delta = \langle D, W \rangle$).
Hence, γ is in all extensions of $\Delta' := \langle D, W \cup \{\alpha\}\rangle$.
The definition of extensions is invariant under replacing any formula by an equivalent formula.
Thus, $\langle D, W \cup \{\beta\}\rangle$ has exactly the same extensions as Δ', and γ is in every one of them. Hence, $\beta \mid\!\sim_\Delta \gamma$. □

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Desirable properties: Right Weakening

Right Weakening (RW):

$$\frac{\models \alpha \to \beta,\ \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}$$

- *Rationale*: If something can be concluded normally, then everything classically implied should also be concluded normally.
- *Example*: Assume that
  `Mary goes` normally implies `Clive goes` and `John goes`.
  Then we would expect that
  `Mary goes` normally implies `Clive goes`.
- From (Ref) & (RW) Supraclassicality follows:

$$\alpha \mid\!\sim \alpha \quad + \quad \frac{\models \alpha \to \beta,\ \alpha \mid\!\sim \alpha}{\alpha \mid\!\sim \beta} \quad \Longrightarrow \quad \frac{\alpha \models \beta}{\alpha \mid\!\sim \beta}$$

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Right Weakening in default logic

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Default logic satisfies Right Weakening.

Proof.

Assume $\models \alpha \rightarrow \beta$ and $\gamma \mid\!\sim_\Delta \alpha$ (with $\Delta = \langle D, W \rangle$).

Hence, α is in each extension E of the default theory $\langle D, W \cup \{\gamma\} \rangle$.

Since extensions are closed under logical consequence, β must also be in each extension of $\langle D, W \cup \{\gamma\} \rangle$.

Hence, $\gamma \mid\!\sim_\Delta \beta$ $\square$

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Desirable properties: Cut

Cut:

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

- *Rationale*: If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.
- *Example*: Assume that
  `John goes` normally implies `Mary goes.`
  Assume further that
  `John goes` and `Mary goes` normally implies `Clive goes.`
  Then we would expect that
  `John goes` normally implies `Clive goes.`

Cut in default logic

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Proposition

Default logic satisfies Cut.

Proof idea.

Assume $\alpha \mid\!\sim_\Delta \beta$ (with $\Delta = \langle D, W \rangle$). Hence β is contained in each extension of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$. Show that every extension E of Δ' is also an extension of $\Delta'' = \langle D, W \cup \{\alpha \wedge \beta\} \rangle$.

- Consistency of justifications of defaults is tested against E both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \wedge \beta\}$ case.
- The preconditions that are derivable when starting from $W \cup \{\alpha\}$ are also derivable when starting from $W \cup \{\alpha \wedge \beta\}$.
- $W \cup \{\alpha \wedge \beta\}$ does not allow for deriving further preconditions because also in the $W \cup \{\alpha\}$ case at some point β is derived.

Hence, because γ belongs to all extensions of Δ'' ($\alpha \wedge \beta \mid\!\sim \gamma$), it also belongs to all extensions of Δ' ($\alpha \mid\!\sim \gamma$). □

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Desirable properties: Cautious Monotonicity

Cautious Monotonicity (CM):

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$

- *Rationale*: In general, adding new premises may cancel some conclusions.
  However, existing conclusions may be added to the premises without canceling any conclusions!
- *Example*: Assume that
  `Mary goes` normally implies `Clive goes` and
  `Mary goes` normally implies `John goes`.
  `Mary goes` and `Jack goes` might not normally imply that `John goes`.
  However, `Mary goes` and `Clive goes` should normally imply that `John goes`.

Cautious Monotonicity in default logic

Proposition

Default logic does not satisfy Cautious Monotonicity.

Proof.

Consider the default theory $\langle D, W\rangle$ with

$$D = \left\{ \frac{a : g}{g}, \frac{g : b}{b}, \frac{b : \neg g}{\neg g} \right\} \text{ and } W = \{a\}.$$

$E = \mathrm{Th}(\{a,b,g\})$ is the only extension of $\langle D, W\rangle$ and thus both b and g follow skeptically (i.e., we have $a \mid\sim_{\langle D,\emptyset\rangle} b$ and $a \mid\sim_{\langle D,\emptyset\rangle} g$).

For $\langle D, \{a \wedge b\}\rangle$ also $\mathrm{Th}(\{a,b,\neg g\})$ is an extension, and thus g does not follow skeptically (i.e., $a \wedge b \not\mid\sim_{\langle D,\emptyset\rangle} g$). □

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Cumulativity

Lemma

Rules (Cut) & (CM) can be equivalently stated as follows:

> *If $\alpha \mid\!\sim \beta$, then the sets of plausible conclusions from α and $\alpha \wedge \beta$ are identical.*

This property is called Cumulativity.

Proof.

$\Rightarrow$: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \mid\!\sim \beta$.
Assume further that $\alpha \mid\!\sim \gamma$. By applying (CM), we obtain $\alpha \wedge \beta \mid\!\sim \gamma$.
Similarly, by applying (Cut), from $\alpha \wedge \beta \mid\!\sim \gamma$ it follows $\alpha \mid\!\sim \gamma$.
Hence the plausible conclusions from α and $\alpha \wedge \beta$ are the same.
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4. Cut

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

5. Cautious Monotonicity

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Derived rules in C

- Equivalence:

$$\frac{\alpha \mid\!\sim \beta,\ \beta \mid\!\sim \alpha,\ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$$

- And:

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$$

- MPC:

$$\frac{\alpha \mid\!\sim \beta \rightarrow \gamma,\ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$

Derived rules: proofs

Proof (Equivalence).

Assumption:	$\alpha \mid\!\sim \beta, \quad \beta \mid\!\sim \alpha, \quad \alpha \mid\!\sim \gamma$
Cautious Monotonicity:	$\alpha \wedge \beta \mid\!\sim \gamma$
Left L Equivalence:	$\beta \wedge \alpha \mid\!\sim \gamma$
Cut:	$\beta \mid\!\sim \gamma$

□

Proof (And).

Assumption:	$\alpha \mid\!\sim \beta, \quad \alpha \mid\!\sim \gamma$
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Supraclassicality:	$\alpha \wedge \beta \wedge \gamma \mid\!\sim \beta \wedge \gamma$
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MPC is an exercise.

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Assumption:	$\alpha \mid\sim \beta, \quad \beta \mid\sim \alpha, \quad \alpha \mid\sim \gamma$
Cautious Monotonicity:	$\alpha \wedge \beta \mid\sim \gamma$
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Cut:	$\beta \mid\sim \gamma$

□

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MPC is an exercise.

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MPC is an exercise.

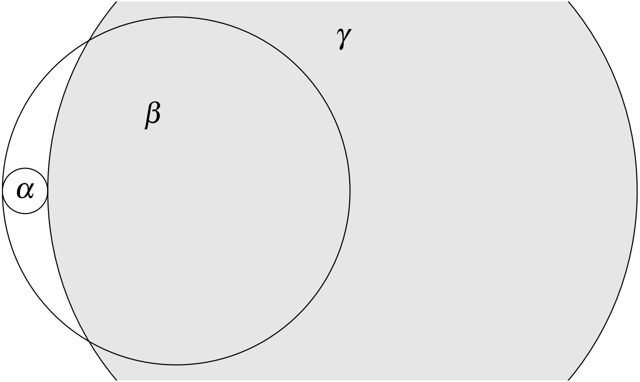
Undesirable properties: Monotonicity and Contraposition

- Monotonicity:
$$\frac{\models \alpha \rightarrow \beta,\ \beta \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \gamma}$$
  - *Example*: Let us assume that
    `John goes` normally implies `Mary goes`.
    Now we will probably not expect that
    `John goes` and `Joan` (who is not in talking terms with Mary) `goes` normally implies `Mary goes`.

- Contraposition:
$$\frac{\alpha \mathrel{|\!\sim} \beta}{\neg\beta \mathrel{|\!\sim} \neg\alpha}$$
  - *Example*: Let us assume that
    `John goes` normally implies `Mary goes`.
    Would we expect that
    `Mary does not go` normally implies `John does not go`?
    What if `John goes` always?

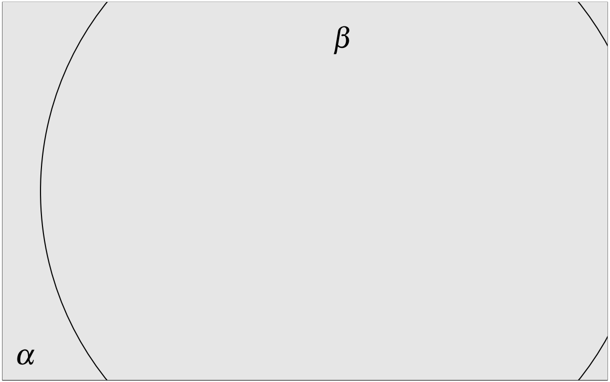
Undesirable properties: Monotonicity

$\alpha \models \beta$, $\beta \mid\!\sim \gamma$, but not $\alpha \mid\!\sim \gamma$ — pictorially:



Undesirable properties: Contraposition

$\alpha \mathrel{|\!\sim} \beta$, but not $\neg\beta \mathrel{|\!\sim} \neg\alpha$ — pictorially:



Undesirable properties: Transitivity & EHD

- Transitivity:

$$\frac{\alpha \mid\!\sim \beta,\ \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

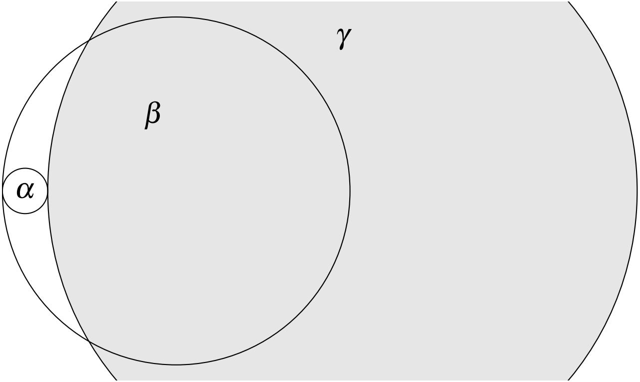
  - *Example*: Let us assume that `John goes` normally implies `Mary goes` and `Mary goes` normally implies `Jack goes`.
    Now, should `John goes` normally imply that `Jack goes`?
    What, if `John goes` very seldom?

- Easy Half of the Deduction Theorem (EHD):

$$\frac{\alpha \mid\!\sim \beta \rightarrow \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$

Undesirable properties: Transitivity

$\alpha \mid\!\sim \beta$, $\beta \mid\!\sim \gamma$, but not $\alpha \mid\!\sim \gamma$ — pictorially:



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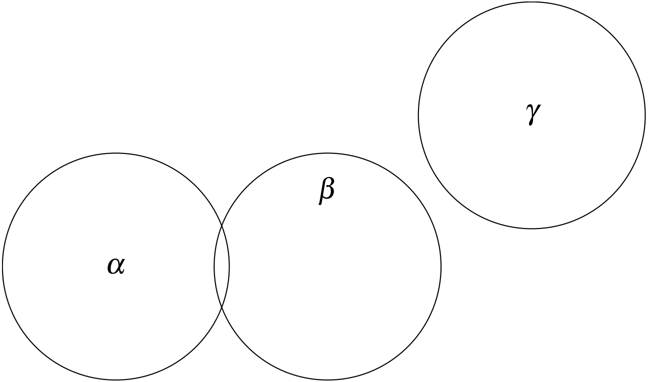

Monotonicity vs EHD

Theorem

*In the presence of the rules in system **C**, the rules Monotonicity and EHD are equivalent.*

Proof.

Monotonicity $\Rightarrow$ EHD:

- $\alpha \mathrel{|\!\sim} \beta \rightarrow \gamma$ (assumption)
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Monotonicity $\Leftarrow$ Transitivity:

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Monotonicity vs Transitivity

Theorem

*In the presence of the rules in system **C**, the rules Monotonicity and Transitivity are equivalent.*

Proof.

Monotonicity $\Rightarrow$ Transitivity:

- $\alpha \mid\!\sim \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\alpha \wedge \beta \mid\!\sim \gamma$ (Monotonicity)
- $\alpha \mid\!\sim \gamma$ (Cut)

Monotonicity $\Leftarrow$ Transitivity:

- $\models \alpha \rightarrow \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \mid\!\sim \beta$ (Supraclassicality)
- $\alpha \mid\!\sim \gamma$ (Transitivity)

□

Contraposition?

Theorem

In the presence of Right Weakening, Contraposition implies Monotonicity.

Proof.

- $\models \alpha \rightarrow \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\neg\gamma \mid\!\sim \neg\beta$ (Contraposition)
- $\models \neg\beta \rightarrow \neg\alpha$ (classical contraposition)
- $\neg\gamma \mid\!\sim \neg\alpha$ (RW)
- $\alpha \mid\!\sim \gamma$ (Contraposition) □

Note: Monotonicity does not imply Contraposition, even in the presence of all rules of system **C**!

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Reasoning

Reasoning with conditionals

- How do we reason with $\mid\!\sim$ from φ to ψ?
- *Assumption*: We have some (finite) set K of conditional statements of the form $\alpha \mid\!\sim \beta$.
  The question is: Assuming the statements in K, is it plausible to conclude ψ given φ?
- *Idea*: We consider all cumulative consequence relations that contain K.
  Cumulative consequence relation: any relation $\mid\!\sim$ between propositional logic formulae that is closed unter the rules of system **C**.
- Remark: It suffices to consider only the minimal cumulative consequence relation containing K …

Cumulative closure

Lemma

The set of cumulative consequence relations is closed under (arbitrary) intersections.

Proof.

Let $\mid\!\sim_1$ and $\mid\!\sim_2$ be cumulative consequence relations. We have to show that $\mid\!\sim_1 \cap \mid\!\sim_2$ is a cumulative consequence relation, that is, it is closed under all the rules of system **C**.
Take any instance of any of the rules. If the preconditions are satisfied by $\mid\!\sim_1$ and $\mid\!\sim_2$, then the consequence is trivially also satisfied by both. A similar argument works if we consider an arbitrary family of consequence relations. □

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Theorem

For each set of conditional statements K, there exists a unique minimal cumulative consequence relation containing K.

Proof.

From the previous lemma it is clear that the intersection of all the cumulative consequence relations containing K is already such a cumulative consequence relation.
Obviously, there cannot be two distinct such minimal relations. □

This relation is called the cumulative closure K^C of K.

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Semantics

Cumulative models – informally

- We will now try to characterize cumulative reasoning model-theoretically.
- *Idea*: Cumulative models consist of states ordered by a preference relation.
- States characterize beliefs.
- The preference relation, $\prec$, expresses the normality of the beliefs.
  We read $s \prec t$ as: state s is preferred to/more normal than state t.
- We say: $\alpha \mid\sim \beta$ is accepted in a model if in all most preferred states in which α is true also β is true.

Preference relation

We consider an arbitrary binary relation $\prec$ on a given set of states S.

Later, we will assume that $\prec$ has particular properties, e.g., that $\prec$ is irreflexive, asymmetric, transitive, a partial order, ...

... but currently we make no such restrictions.

We need a condition on state sets claiming that each state is, or is related to, a most preferred state.

Definition (Smoothness)

Let $P \subseteq S$.

- We say that $s \in P$ is minimal in P if $s' \not\prec s$ for each $s' \in P$.
- P is called smooth if for each $s \in P$, either s is minimal in P or there exists an s' such that s' is minimal in P and $s' \prec s$.

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Cumulative Models – formally

Let $\mathcal{U}$ be the set of all possible worlds (i.e., propositional interpretations).

- A cumulative model is a triple $W = \langle S, l, \prec \rangle$ such that
  1. S is a set of states,
  2. l is a mapping $l : S \to 2^{\mathcal{U}}$, and
  3. $\prec$ is an arbitrary binary relation on S

  such that the smoothness condition is satisfied (see below).
- A state $s \in S$ satisfies a formula α ($s \mid\!\equiv \alpha$) if $m \models \alpha$ for each propositional interpretation $m \in l(s)$.
  The set of states satisfying α is denoted by $\widehat{\alpha}$.
- Smoothness condition: A cumulative model satisfies this condition if for all formulae α, $\widehat{\alpha}$ is smooth.

Consequence relation induced by a cumulative model

A cumulative model W induces a consequence relation $\mid\!\sim_W$ as follows:

$$\alpha \mid\!\sim_W \beta \text{ iff } s \models\!\!\!\equiv \beta \text{ for every minimal } s \text{ in } \widehat{\alpha}.$$

Example

Model $W = \langle \{s_1, s_2, s_3\}, l, \prec \rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

$$l(s_1) = \{\{\neg p, b, f\}\}$$
$$l(s_2) = \{\{p, b, \neg f\}\}$$
$$l(s_3) = \{\{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}\}$$

Does W satisfy the smoothness condition?

$\neg p \wedge \neg b \mid\!\sim f$? N Also: $\neg p \wedge \neg b \not\mid\!\sim \neg f$!

$p \mid\!\sim \neg f$? Y

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Soundness 1

Theorem

If W is a cumulative model, then $\mid\!\sim_W$ is a cumulative consequence relation.

Proof.

- Reflexivity: satisfied.
- LLE: satisfied.
- RW: satisfied.
- Cut: $\alpha \mid\!\sim_W \beta, \alpha \wedge \beta \mid\!\sim_W \gamma \Rightarrow \alpha \mid\!\sim_W \gamma$. Assume that all minimal elements of $\widehat{\alpha}$ satisfy β, and all minimal elements of $\widehat{\alpha \wedge \beta}$ satisfy γ. Every minimal element of $\widehat{\alpha}$ satisfies $\alpha \wedge \beta$. Since $\widehat{\alpha \wedge \beta} \subseteq \widehat{\alpha}$, all minimal elements of $\widehat{\alpha}$ are also minimal elements of $\widehat{\alpha \wedge \beta}$. Hence $\alpha \mid\!\sim_W \gamma$.

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Soundness 2

Proof continues...

- Cautious Monotonicity: $(\alpha \mid\!\sim \beta, \alpha \mid\!\sim \gamma \Rightarrow \alpha \wedge \beta \mid\!\sim \gamma)$

  Assume $\alpha \mid\!\sim_W \beta$ and $\alpha \mid\!\sim_W \gamma$. We have to show: $\alpha \wedge \beta \mid\!\sim_W \gamma$, i.e., $s \models\!\!\equiv \gamma$ for all minimal $s \in \widehat{\alpha \wedge \beta}$.

  Clearly, every minimal $s \in \widehat{\alpha \wedge \beta}$ is in $\widehat{\alpha}$.

  We show that every minimal $s \in \widehat{\alpha \wedge \beta}$ is minimal in $\widehat{\alpha}$.

  Assumption: There is s that is minimal in $\widehat{\alpha \wedge \beta}$, but not minimal in $\widehat{\alpha}$. Because of smoothness there is minimal $s' \in \widehat{\alpha}$ such that $s' \prec s$. We know, however, that $s' \models\!\!\equiv \beta$, which means that $s' \in \widehat{\alpha \wedge \beta}$. Hence s is not minimal in $\widehat{\alpha \wedge \beta}$. Contradiction!

  Hence s must be minimal in $\widehat{\alpha}$, and therefore $s \models\!\!\equiv \gamma$. Because this is true for all minimal elements in $\widehat{\alpha \wedge \beta}$, we get $\alpha \wedge \beta \mid\!\sim_W \gamma$. $\square$

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Soundness 2

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Consequence: counterexamples

Now we have a method for showing that a principle does not hold for cumulative consequence relations:
… construct a cumulative model that falsifies the principle.

Contraposition: $\alpha \mid\sim \beta \Rightarrow \neg\beta \mid\sim \neg\alpha$

$$
\begin{aligned}
W &= \langle S, l, \prec \rangle \\
S &= \{s_1, s_2\} \\
& s_i \not\prec s_j \ \forall s_i, s_j \in S \\
l(s_1) &= \{\{a, b\}\} \\
l(s_2) &= \{\{a, \neg b\}, \{\neg a, \neg b\}\}
\end{aligned}
$$

W is a cumulative model with $a \mid\sim_W b$, but $\neg b \not\mid\sim_W \neg a$.

Completeness?

- Each cumulative model W induces a cumulative consequence relation $\mid\!\sim_W$.
- Problem: Can we generate all cumulative consequence relations in this way?
- We can! There is a representation theorem:

Theorem (Representation of cumulative consequence)

A consequence relation is cumulative if and only if it is induced by some cumulative model.

$\rightsquigarrow$ Cumulative consequence can be characterized independently from the set of inference rules.

Transitivity of the preference relation?

- Could we strengthen the preference relation to transitive relations without sacrificing anything?
  No!
- In such models, the following additional principle called Loop is valid:

$$\frac{\alpha_0 \mathrel{|\!\sim} \alpha_1, \alpha_1 \mathrel{|\!\sim} \alpha_2, \ldots, \alpha_k \mathrel{|\!\sim} \alpha_0}{\alpha_0 \mathrel{|\!\sim} \alpha_k}$$

- For the system **CL** $=$ **C** $+$ (Loop) and cumulative models with transitive preference relations, we could prove another representation theorem.

The Or Rule

Or rule:

$$\frac{\alpha \mid\!\sim \gamma,\ \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}$$

Not valid in system **C**. Counterexample:

$$\begin{aligned} W &= \langle S, l, \prec \rangle \\ S &= \{s_1, s_2, s_3\}, s_i \not\prec s_j\ \forall s_i, s_j \in S \\ l(s_1) &= \{\{a,b,c\},\{a,\neg b,c\}\} \\ l(s_2) &= \{\{a,b,c\},\{\neg a,b,c\}\} \\ l(s_3) &= \{\{a,b,\neg c\},\{a,\neg b,\neg c\},\{\neg a,b,\neg c\}\} \end{aligned}$$

$a \mid\!\sim_W c$, $b \mid\!\sim_W c$, but not $a \vee b \mid\!\sim_W c$.

Note: Or is not valid in default logic.

Preferential Reasoning

System P

- System **P** contains all rules of **C** and the Or rule.
- A consequence relation that satisfies **P** is called preferential.
- Derived rules in **P**:
  - Hard half of the deduction theorem (S):

$$\frac{\alpha \wedge \beta \mid\sim \gamma}{\alpha \mid\sim \beta \rightarrow \gamma}$$

  - Proof by case analysis (D):

$$\frac{\alpha \wedge \neg\beta \mid\sim \gamma,\ \alpha \wedge \beta \mid\sim \gamma}{\alpha \mid\sim \gamma}$$

- D and Or are equivalent in the presence of the rules in C.

Preferential models

Definition

A cumulative model $W = \langle S, l, \prec \rangle$ such that $\prec$ is a strict partial order (irreflexive and transitive) and $|l(s)| = 1$ for all $s \in S$ is called a preferential model.

Theorem (Soundness)

The consequence relation $\mid\!\sim_W$ induced by a preferential model is preferential.

Proof.

Since W is cumulative, we only have to verify that Or holds. Note that in preferential models we have $\widehat{\alpha \vee \beta} = \widehat{\alpha} \cup \widehat{\beta}$. Suppose $\alpha \mid\!\sim_W \gamma$ and $\beta \mid\!\sim_W \gamma$. Because of the above equation, each minimal state of $\widehat{\alpha \vee \beta}$ is minimal in $\widehat{\alpha} \cup \widehat{\beta}$. Since γ is satisfied in all minimal states in $\widehat{\alpha} \cup \widehat{\beta}$, γ is also satisfied in all minimal states of $\widehat{\alpha \vee \beta}$. Hence $\alpha \vee \beta \mid\!\sim_W \gamma$. $\square$

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Theorem (Representation of preferential consequence)

A consequence relation is preferential if and only if it is induced by a preferential model.

Proof.

Similar to the one for **C**. □

Summary of cumulative systems

System	Models
C	
Reflexivity	States: sets of worlds
Left Logical Equivalence	Preference relation: arbitrary
Right Weakening	Models must be smooth
Cut	
Cautious Monotonicity	
CL	
+ Loop	Preference relation: strict partial order
P	
+ Or	States: singletons

Strengthening the consequence relation

- System **C** and System **P** do not produce many of the inferences one would hope for:

  Given $K = \{Bird \mid\!\sim Flies\}$ one cannot conclude $Red \land Bird \mid\!\sim Flies$!

- In general, adding information that is irrelevant cancels the plausible conclusions.
  $\Longrightarrow$ Cumulative and Preferential consequence relations are too nonmonotonic.

- The plausible conclusions have to be strengthened!

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Strengthening the consequence relations

- The rules so far seem to be reasonable: one cannot think of rules of the same form (if we have some plausible implications, other plausible implications should hold) that could be added.
- However, there are other types of rules one might want add.
- Disjunctive Rationality:

$$\frac{\alpha \not\mid\!\sim \gamma \,,\, \beta \not\mid\!\sim \gamma}{\alpha \vee \beta \not\mid\!\sim \gamma}$$

- Rational Monotonicity:

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- Note: Consequence relations obeying these rules are not closed under intersection, which is a problem.

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Summary

- Instead of ad hoc extensions of the logical machinery, analyze the properties of nonmonotonic consequence relations.
- Correspondence between rule system and models for System **C**, and for System **P** could also be established wrt. a probabilistic semantics.
- Irrelevant information poses a problem. Solution approaches: rational monotonicity, maximum entropy approach.

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