Principles of Knowledge Representation and Reasoning

Nonmonotonic Reasoning

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

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Introduction

A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *She has an essay to write.*

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What do you conclude?

A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *She has an essay to write.*

In empirical studies 95% of all subjects conclude (modus ponens):

- *She will study late in the library.*

A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *If the library is open, she will study late in the library.*
- *She has an essay to write.*

What do you conclude now?

A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *If the library is open, she will study late in the library.*
- *She has an essay to write.*

In cognitive studies now only 60% of the subjects conclude:

- *She will study late in the library.*

A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *She has an essay to write.*

Conclusion?

- *She will study late in the library.*

Reasoning tasks like this (suppression task; Byrne, 1989) suggest that humans often do reason as suggested by classical logics

Nonmonotonic reasoning

- All logics presented so far are monotonic.
- A logic is called monotonic if all (logical) conclusions from a knowledge base remain justified when new information is added to the knowledge base.
- Cognitive studies indicate that everyday reasoning is often nonmonotonic (Stenning & Lambalgen, 2008; Johnson-Laird, 2010, etc.).
- When humans reason they use:
  - rules that may have exceptions:

    If Mary has an essay to write, she normally will study late in the library.
  - default assumptions:

    The library is open.

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Defaults in knowledge bases

Often we use default assumptions when definite information is not available or when we want to fix a standard value:

1. employee(anne)
2. employee(bert)
3. employee(carla)
4. employee(detlef)
5. employee(thomas)
6. onUnpaidMPaternityLeave(thomas)
7. employee(X) $\wedge\neg$ onUnpaidMPaternityLeave(X) $\rightarrow$ gettingSalary(X)
8. Typically: employee(X) $\rightarrow\neg$ onUnpaidMPaternityLeave(X)

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Defaults in common sense reasoning

1. Tweety is a bird like other birds.
2. During the summer he stays in Northern Europe, in the winter he stays in Africa.

- Would you expect Tweety to be able to fly?
- How does Tweety get from Northern Europe to Africa?

How would you formalize this in formal logic so that you get the expected answers?

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A formalization …

1. bird(tweety)
2. spend-summer(tweety, northern-europe) $\wedge$ spend-winter(tweety, africa)
3. $\forall x(\text{bird}(x) \rightarrow \text{can-fly}(x))$
4. far-away(northern-europe, africa)
5. $\forall xyz(\text{can-fly}(x) \wedge \text{far-away}(y,z) \wedge \text{spend-summer}(x,y) \wedge \text{spend-winter}(x,z) \rightarrow \text{flies}(x,y,z))$

- But: The implication (3) is just a reasonable assumption.
- What if Tweety is an emu?

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Examples of such reasoning patterns

Closed world assumption: Database of ground atoms. All ground atoms not present are assumed to be false.

Negation as failure: In PROLOG, NOT(P) means "*P is not provable*" instead of "*P is provably false*".

Non-strict inheritance: An attribute value is inherited only if there is no more specialized information contradicting the attribute value.

Reasoning about actions: When reasoning about actions, it is usually assumed that a property changes only if it has to change, i.e., properties by default do not change.

Default, defeasible, and nonmonotonic reasoning

Default reasoning: Jump to a conclusion if there is no information that contradicts the conclusion.

Defeasible reasoning: Reasoning based on assumptions that can turn out to be wrong: conclusions are defeasible. In particular, default reasoning is defeasible.

Nonmonotonic reasoning: In classical logic, the set of consequences grows monotonically with the set of premises. If reasoning is defeasible, then reasoning becomes nonmonotonic.

Approaches to nonmonotonic reasoning

- Consistency-based: Extend classical theory by rules that test whether an assumption is consistent with existing beliefs

$\Rightarrow$ Nonmonotonic logics such as DL (default logic), NMLP (nonmonotonic logic programming)

- Entailment-based on normal models: Models are ordered by normality. Entailment is determined by considering the most normal models only.

$\Rightarrow$ Circumscription, preferential and cumulative logics

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NM Logic – Consistency-based

If φ typically implies ψ, φ is given, and it is consistent to assume ψ, then conclude ψ.

1. Typically bird(x) implies can-fly(x)
2. $\forall x(\text{emu}(x) \rightarrow \text{bird}(x))$
3. $\forall x(\text{emu}(x) \rightarrow \neg\text{can-fly}(x))$
4. bird(tweety)

$\Rightarrow$ can-fly(tweety)

5. ... + emu(tweety)

$\Rightarrow \neg$ can-fly(tweety)

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NM Logic – Normal models

If φ typically implies ψ, then the models satisfying $\varphi \wedge \psi$ should be more normal than those satisfying $\varphi \wedge \neg\psi$.

Similar idea: try to minimize the interpretation of "Abnormality" predicates.

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Default Logic

Default Logic – Outline

Reiter's default logic: motivation

- We want to express something like "typically birds fly".
- Add non-logical inference rule

$$\frac{\text{bird}(x) : \text{can-fly}(x)}{\text{can-fly}(x)}$$

with the intended meaning:
If x is a bird and if it is consistent to assume that x can fly, then conclude that x can fly.

- Exceptions can be represented as formulae:

$$\forall x(\text{penguin}(x) \rightarrow \neg\text{can-fly}(x))$$
$$\forall x(\text{emu}(x) \rightarrow \neg\text{can-fly}(x))$$
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Formal framework

- FOL with classical provability relation $\vdash$ and deductive closure: $\mathrm{Th}(\Phi) := \{\varphi | \Phi \vdash \varphi\}$
- Default rules: $\dfrac{\alpha : \beta}{\gamma}$
  - α: Prerequisite: must have been derived before rule can be applied.
  - β: Consistency condition: the negation may not be derivable.
  - γ: Consequence: will be concluded.
- A default rule is closed if it does not contain free variables.
- (Closed) default theory: A pair $\langle D, W \rangle$, where D is a countable set of (closed) default rules and W is a countable set of FOL formulae.

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Extensions of default theories

Default theories extend the theory given by W using the default rules in D ($\rightsquigarrow$ extensions). There may be zero, one, or many extensions.

Example

$$W = \{a, \neg b \vee \neg c\}$$

$$D = \left\{ \frac{a : b}{b}, \frac{a : c}{c} \right\}$$

One extension contains b, the other contains c.

Intuitively, an extension is a set of beliefs resulting from W and D.

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Decision problems about extensions in default logic

Existence of extensions: Does a default theory have an extension?

Credulous reasoning: If φ is in at least one extension, φ is a credulous default conclusion.

Skeptical reasoning: If φ is in all extensions, φ is a skeptical default conclusion.

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Extensions (informally)

Desirable properties of an extension E of $\langle D, W \rangle$:

1. Contains all facts: $W \subseteq E$.
2. Is deductively closed: $E = \mathrm{Th}(E)$.
3. All applicable default rules have been applied:
   If
   1. $(\frac{\alpha:\beta}{\gamma}) \in D$,
   2. $\alpha \in E$,
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   then $\gamma \in E$.

- Further requirement: Application of default rules must follow in sequence (groundedness).

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Groundedness

Example

$$W = \emptyset$$

$$D = \left\{ \frac{a : b}{b}, \frac{b : a}{a} \right\}$$

Question: Should $\mathrm{Th}(\{a, b\})$ be an extension?

Answer: No!

a can only be derived if we already have derived b.

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Extensions (formally)

Definition

Let $\Delta = \langle D, W \rangle$ be a closed default theory.
Let E be any set of closed formulae.
Define:

$$E_0 = W$$

$$E_i = \mathrm{Th}(E_{i-1}) \cup \left\{ \gamma \,\middle|\, \frac{\alpha : \beta}{\gamma} \in D, \alpha \in E_{i-1}, \neg\beta \notin E \right\}$$

E is called an extension of Δ if

$$E = \bigcup_{i=0}^{\infty} E_i .$$

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$$E = \bigcup_{i=0}^{\infty} E_i.$$

How to use this definition?

- The definition does not tell us how to construct an extension.
- However, it tells us how to check whether a set is an extension:
  1. Guess a set E.
  2. Then construct sets E_i by starting with W.
  3. If $E = \bigcup_{i=0}^{\infty} E_i$, then E is an extension of $\langle D, W \rangle$.

Examples

$$D = \left\{ \frac{a : b}{b}, \frac{b : a}{a} \right\} \qquad W = \{a \vee b\}$$

$$D = \left\{ \frac{a : b}{\neg b} \right\} \qquad W = \emptyset$$

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$$D = \left\{ \frac{: a}{a}, \frac{: b}{b}, \frac{: c}{c} \right\} \qquad W = \{b \rightarrow \neg a \wedge \neg c\}$$

$$D = \left\{ \frac{: c}{\neg d}, \frac{: d}{\neg e}, \frac{: e}{\neg f} \right\} \qquad W = \emptyset$$

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$$D = \left\{ \frac{a : b}{c}, \frac{a : d}{e} \right\} \qquad W = \{a, \neg b \vee \neg d\}$$

Questions, questions, questions ...

- What can we say about the existence of extensions?
- How are the different extensions related to each other?
  - Can one extension be a subset of another one?
  - Are extensions pairwise incompatible (i.e. jointly inconsistent)?
- Can an extension be inconsistent?

Properties of extensions: existence

Theorem

1. *If W is inconsistent, there is only one extension.*
2. *A closed default theory $\langle D, W \rangle$ has an inconsistent extensions E if and only if W is inconsistent.*

Proof idea.

1. If W is inconsistent, no default rule is applicable and $\mathrm{Th}(W)$ is the only extension (which is inconsistent as well).
2. Claim 1 $\Longrightarrow$ the **if**-part.
   For **only if**: Let W be consistent and assume that there exists an inconsistent extension E.
   Then there exists a consistent E_i such that E_{i+1} is inconsistent.
   That is, there is at least one applied default $\alpha_i : \beta_i \,/\, \gamma_i$ with $\gamma_i \in E_{i+1} \setminus \mathrm{Th}(E_i)$, $\alpha_i \in E_i$, and $\neg\beta_i \notin E$.
   But this contradicts the inconsistency of E. □

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   But this contradicts the inconsistency of E. $\square$

Properties of extensions

Theorem

If E and F are extensions of $\langle D, W \rangle$ such that $E \subseteq F$, then $E = F$.

Proof sketch.

$E = \bigcup_{i=0}^{\infty} E_i$ and $F = \bigcup_{i=0}^{\infty} F_i$. Use induction to show $F_i \subseteq E_i$.

Base case $i = 0$: Trivially $E_0 = F_0 = W$.

Inductive case $i \geq 1$: Assume $\gamma \in F_{i+1}$. Two cases:

1. $\gamma \in \mathrm{Th}(F_i)$ implies $\gamma \in \mathrm{Th}(E_i)$ (because $F_i \subseteq E_i$ by IH), and therefore $\gamma \in E_{i+1}$.
2. Otherwise $\frac{\alpha : \beta}{\gamma} \in D$, $\alpha \in F_i$, $\neg\beta \notin F$. However, then we have $\alpha \in E_i$ (because $F_i \subseteq E_i$) and $\neg\beta \notin E$ (because of $E \subseteq F$), i.e., $\gamma \in E_{i+1}$. □

Introduction

Default Logic

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Special Kinds of Defaults

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Normal default theories

All defaults in a normal default theory are normal:

$$\frac{\alpha : \beta}{\beta}.$$

Theorem

Normal default theories have at least one extension.

Proof sketch.

If W inconsistent, trivial. Otherwise construct

$$\begin{aligned} E_0 &= W \\ E_{i+1} &= \mathrm{Th}(E_i) \cup T_i \end{aligned} \qquad\qquad E = \bigcup_{i=0}^{\infty} E_i$$

where T_i is a maximal set s.t. (1) $E_i \cup T_i$ is consistent and (2) if $\beta \in T_i$ then there is $\frac{\alpha : \beta}{\beta} \in D$ and $\alpha \in E_i$.

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Normal default theories

All defaults in a normal default theory are normal:

$$\frac{\alpha : \beta}{\beta}.$$

Theorem

Normal default theories have at least one extension.

Proof sketch.

If W inconsistent, trivial. Otherwise construct

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Normal default theories: extensions are orthogonal

Theorem (Orthogonality)

Let E and F be distinct extensions of a normal default theory. Then $E \cup F$ is inconsistent.

Proof.

Let $E = \bigcup E_i$ and $F = \bigcup F_i$ with

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Default proofs in normal default theories

Definition

A default proof of γ in a normal default theory $\langle D, W\rangle$ is a finite sequence of defaults $(\delta_i = \frac{\alpha_i : \beta_i}{\beta_i})_{i=1,\ldots,n}$ in D such that

1. $W \cup \{\beta_1, \ldots, \beta_n\} \vdash \gamma$,
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Theorem

Let $\Delta = \langle D, W\rangle$ be a normal default theory so that W is consistent. Then γ has a default proof in Δ if and only if there exists an extension E of Δ such that $\gamma \in E$.

Test 2 (consistency) in the proof procedure suggests that default provability is not even semi-decidable.

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Decidability

Theorem

It is not semi-decidable to test whether a formula follows (skeptically or credulously) from a default theory.

Proof.

Let $\langle D, W \rangle$ be a default theory with $W = \emptyset$ and $D = \left\{ \frac{:\beta}{\beta} \right\}$ with β an arbitrary closed FOL formula. Clearly, β is in some/all extensions of $\langle D, W \rangle$ if and only if β is satisfiable.

The existence of a semi-decision procedure for default proofs implies that there is a semi-decision procedure for satisfiability in FOL.

But this is not possible because FOL validity is semi-decidable and this together with semi-decidability of FOL satisfiability would imply decidability of FOL, which is not the case. □

Complexity of Default Logic

Propositional default logic

- Propositional DL is decidable.
- How difficult is reasoning in propositional DL?
- The skeptical default reasoning problem (does φ follow from Δ skeptically: $\Delta \mid\!\sim \varphi$?) is called PDS, credulous reasoning is called LPDS.
- PDS is coNP-hard: consider $D = \emptyset$, $W = \emptyset$
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Lemma

$PDS \in \Pi_2^p$.

Proof sketch.

We show that the complementary problem UNPDS (is there an extension E such that $\varphi \notin E$) is in Σ_2^p. The algorithm:

1. Guess set $T \subseteq D$ of defaults, those that are applied.
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Π_2^p-Hardness

Lemma

PDS is Π_2^p-hard.

Proof sketch.

Reduction from 2QBF to UNPDS: For $\exists\vec{a}\,\forall\vec{b}\,\varphi(\vec{a},\vec{b})$ with $\vec{a} = a_1,\ldots,a_n$ and $\vec{b} = b_1,\ldots,b_m$ construct $\Delta = \langle D, W\rangle$ with

$$D = \left\{ \frac{:a_i}{a_i}, \frac{:\neg a_i}{\neg a_i}, \frac{:\varphi(\vec{a},\vec{b})}{\varphi(\vec{a},\vec{b})} \right\}, \quad W = \emptyset$$

No extension contains both a_i and $\neg a_i$. Then:

$\Delta \not\mid\!\sim \neg\varphi(\vec{a},\vec{b})$ iff there is an extension E s.t. $\neg\varphi(\vec{a},\vec{b}) \notin E$

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Conclusions & remarks

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LPDS is Σ_2^p-complete, even for defaults of the form $\frac{:\alpha}{\alpha}$.

- PDS is “easier” than reasoning in most modal logics.
- General and normal defaults have the same complexity.
- Polynomial special cases cannot be achieved by restricting, for example, to Horn clauses (satisfiability testing in polynomial time).
- It is necessary to restrict the underlying monotonic reasoning problem and the number of extensions.
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Special Kinds of Defaults

Semi-normal defaults (1)

Semi-normal defaults are sometimes useful:

$$\frac{\alpha : \beta \wedge \gamma}{\beta}$$

Important when one has interacting defaults:

$$\frac{\texttt{Adult}(x) : \texttt{Employed}(x)}{\texttt{Employed}(x)}$$

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For `Student(TOM)` we get two extensions: one with `Employed(TOM)` and the other one with $\neg$`Employed(Tom)`. Since the third rule is "more specific", we may prefer it.

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For `Student(TOM)` we get two extensions: one with `Employed(TOM)` and the other one with $\neg$`Employed(Tom)`.

Since the third rule is "more specific", we may prefer it.

Semi-normal defaults (1)

Semi-normal defaults are sometimes useful:

$$\frac{\alpha : \beta \wedge \gamma}{\beta}$$

Important when one has interacting defaults:

$$\frac{\texttt{Adult}(x) : \texttt{Employed}(x)}{\texttt{Employed}(x)}$$

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For $\texttt{Student(TOM)}$ we get two extensions: one with $\texttt{Employed(TOM)}$ and the other one with $\neg\texttt{Employed(Tom)}$. Since the third rule is "more specific", we may prefer it.

Semi-normal defaults (2)

- Since being a student is an exception, we could use a semi-normal default to exclude students from employed adults:

$$\frac{\mathtt{Student}(x) : \neg\mathtt{Employed}(x)}{\neg\mathtt{Employed}(x)}$$

$$\frac{\mathtt{Adult}(x) : \mathtt{Employed}(x) \wedge \neg\mathtt{Student}(x)}{\mathtt{Employed}(x)}$$

$$\frac{\mathtt{Student}(x) : \mathtt{Adult}(x)}{\mathtt{Adult}(x)}$$

- Representing conflict-resolution by semi-normal defaults becomes clumsy when the number of default rules becomes high.
- A scheme for assigning priorities would be more elegant (there are indeed such schemes).

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Open defaults (1)

- Our examples included open defaults, but the theory covers only closed defaults.
- If we have $\frac{\alpha(\vec{x}):\beta(\vec{x})}{\gamma(\vec{x})}$, then the variables should stand for all nameable objects.
- *Problem*: What about objects that have been introduced implicitly, e.g., via formulae such a $\exists x P(x)$.
- *Solution by Reiter*: Skolemization of all formulae in W and D.
- *Interpretation*: An open default stands for all the closed defaults resulting from substituting ground terms for the variables.

Open defaults (2)

Skolemization can create problems because it preserves satisfiability, but it is not an equivalence transformation.

Example

$\forall x(\mathtt{Man}(x) \leftrightarrow \neg\mathtt{Woman}(x))$
$\forall x(\mathtt{Man}(x) \rightarrow (\exists y(\mathtt{Spouse}(x,y) \wedge \mathtt{Woman}(y)) \vee \mathtt{Bachelor}(x)))$
$\mathtt{Man}(\mathtt{TOM})$
$\mathtt{Spouse}(\mathtt{TOM},\mathtt{MARY})$
$\mathtt{Woman}(\mathtt{MARY})$
$\frac{:\mathtt{Man}(x)}{\mathtt{Man}(x)}$

Skolemization of $\exists y:$ ... enables concluding $\mathtt{Bachelor}(\mathtt{TOM})$! The reason is that for $g(\mathtt{TOM})$ we get $\mathtt{Man}(g(\mathtt{TOM}))$ by default (where g is the Skolem function).

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Open defaults (3)

It is even worse: Logically equivalent theories can have different extensions:

$$W_1 = \{\exists x(P(c,x) \vee Q(c,x))\}$$

$$W_2 = \{\exists x P(c,x) \vee \exists x Q(c,x)\}$$

$$D = \left\{ \frac{P(x,y) \vee Q(x,y) : R}{R} \right\}$$

W_1 and W_2 are logically equivalent. However, the Skolemization of W_1, symbolically $s(W_1)$, is not equivalent with $s(W_2)$. The only extension of $\langle D, W_1 \rangle$ is $\mathrm{Th}(s(W_1) \cup R)$.The only extension of $\langle D, W_2 \rangle$ is $\mathrm{Th}(s(W_2))$.

Note: Skolemization is not the right method to deal with open defaults in the general case.

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Outlook

Although Reiter's definition of DL makes sense, one can come up with a number of variations and extend the investigation …

- Extensions can be defined differently (e.g., by remembering consistency conditions).
- … or by removing the groundedness condition.
- Open defaults can be handled differently (more model-theoretically).
- General proof methods for the finite, decidable case
- Applications of default logic:
  - Diagnosis
  - Reasoning about actions

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