Principles of Knowledge Representation and Reasoning

Modal Logics

Albert-Ludwigs-Universität Freiburg

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Bernhard Nebel, Stefan Wölfl, and Julien Hué

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Motivation

Motivation for studying modal logics

- Notions like believing and knowing require a more general semantics than e.g. propositional logic has.
- Some KR formalisms can be understood as (fragments of) a propositional modal logic.
- Application 1: Spatial representation formalism RCC8
- Application 2: Description logics
- Application 3: Reasoning about time
- Application 4: Reasoning about actions, strategies, etc.

Motivation for modal logics

Often, we want to state something where we have an "embedded proposition":

- John believes that it is Sunday.
- I know that $2^{10} = 1024$.

Reasoning with embedded propositions:

- *John believes that if it is Sunday, then shops are closed.*
- *John believes that it is Sunday.*
- This implies (assuming belief is closed under modus ponens):
  John believes that shops are closed.

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Syntax

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Propositional logic + operators $\Box$ & $\Diamond$ (Box & Diamond):

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\begin{array}{rcll}
\varphi & \longrightarrow & \ldots & \text{classical propositional formula} \\
& | & \Box\varphi' & \text{Box} \\
& | & \Diamond\varphi' & \text{Diamond}
\end{array}
$$

$\Box$ and $\Diamond$ have the same operator precedence as $\neg$.

Some possible readings of $\Box\varphi$:

- Necessarily φ (alethic)
- Always φ (temporal)
- φ should be true (deontic)
- Agent A believes that φ (doxastic)
- Agent A knows that φ (epistemic)

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Semantics

Truth-functional semantics?

- Is it possible to define the meaning of $\Box\varphi$ truth-functionally, i.e. by referring to the truth value of φ only?
- An attempt to interpret necessity truth-functionally:
  - If φ is false, then $\Box\varphi$ should be false.
  - If φ is true, then ...
    - ... $\Box\varphi$ should be true $\leadsto$ $\Box$ is the identity function
    - ... $\Box\varphi$ should be false $\leadsto$ $\Box\varphi$ is identical to falsity
- Note: There are only 4 different unary Boolean functions $\{T,F\} \to \{T,F\}$.

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Semantics: the idea

In classical propositional logic, formulae are interpreted over single interpretations and are evaluated to true or false.

In modal logics one considers sets of interpretations: possible worlds (physically possible, conceivable, ...).

Main idea:

- Consider a world (interpretation) w and a set of worlds W which are possible with respect to w.
- A classical formula (with no modal operators) φ is true with respect to (w, W) iff φ is true in w.
- $\Box\varphi$ is true wrt. (w, W) iff φ is true in all worlds in W.
- $\Diamond\varphi$ is true wrt. (w, W) iff φ is true in some world in W.
- Meanings of $\Box$ and $\Diamond$ are interrelated by: $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

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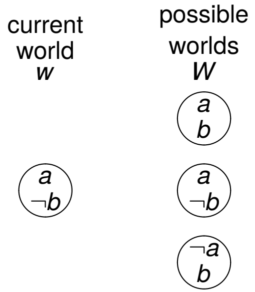

Examples:

- $a \wedge \neg b$ is true relative to (w, W).
- $\Box a$ is not true relative to (w, W).
- $\Box(a \vee b)$ is true relative to (w, W).

Question: How to evaluate modal formulae in $w \in W$?

$\leadsto$ For each world, we specify a set of possible worlds.

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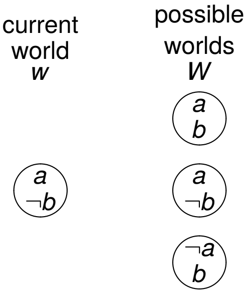

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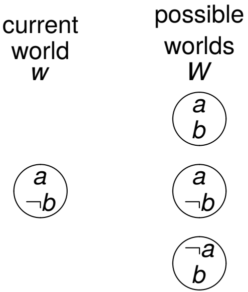

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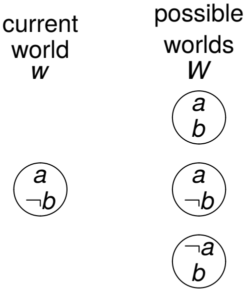

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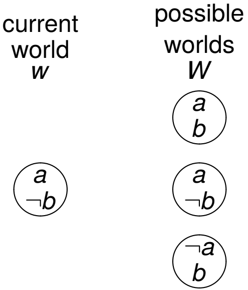

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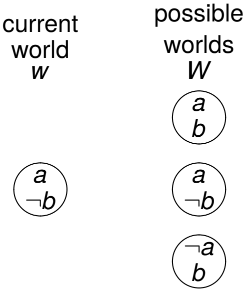

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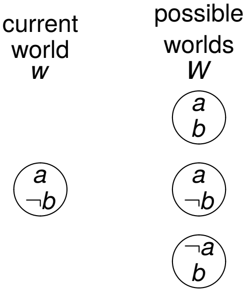

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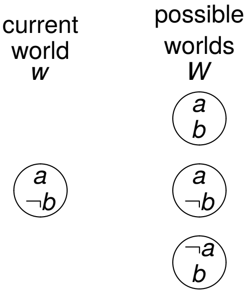

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Frames, interpretations, and worlds

Definition (Kripke frame)

A (Kripke, relational) frame is a pair $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set (of worlds) and $R \subseteq W \times W$ is a binary relation on W (accessibility relation).

For $(w, v) \in R$ we write also $w\,R\,v$. We say that v is an R-successor of w or that v is R-reachable from w.

Definition (Kripke model)

For a given set of propositional variables Σ, a Kripke model (or interpretation) based on the frame $\mathcal{F} = \langle W, R \rangle$ is a triple $\mathcal{I} = \langle W, R, \pi \rangle$, where π is a function that maps worlds w to truth assignments $\pi_w : \Sigma \to \{T, F\}$, i.e.:

$$\pi : W \to \{T, F\}^{\Sigma}, \; w \mapsto \pi_w.$$

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Semantics: truth in a world

A formula φ is true in world w in an interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

$$
\begin{array}{ll}
\mathcal{I}, w \models a & \text{iff } \pi_w(a) = T \\
\mathcal{I}, w \models \top & \\
\mathcal{I}, w \not\models \bot & \\
\mathcal{I}, w \models \neg\varphi & \text{iff } \mathcal{I}, w \not\models \varphi \\
\mathcal{I}, w \models \varphi \wedge \psi & \text{iff } \mathcal{I}, w \models \varphi \text{ and } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \varphi \vee \psi & \text{iff } \mathcal{I}, w \models \varphi \text{ or } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \varphi \rightarrow \psi & \text{iff } \mathcal{I}, w \not\models \varphi \text{ or } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \varphi \leftrightarrow \psi & \text{iff } \mathcal{I}, w \models \varphi \text{ if and only if } \mathcal{I}, w \models \psi \\
\mathcal{I}, w \models \Box\varphi & \text{iff } \mathcal{I}, u \models \varphi, \text{ for all } u \text{ s.t. } wRu \\
\mathcal{I}, w \models \Diamond\varphi & \text{iff } \mathcal{I}, u \models \varphi, \text{ for at least one } u \text{ s.t. } wRu
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Satisfiability and validity

A formula φ is satisfiable in an interpretation $\mathcal{I}$ if there exists a world w in $\mathcal{I}$ such that $\mathcal{I}, w \models \varphi$.

A formula φ is satisfiable in a frame $\mathcal{F}$ (satisfiable in a class of frames $\mathcal{C}$) if it is satisfiable in an interpretation $\mathcal{I}$ based on $\mathcal{F}$ (satisfiable in an interpretation $\mathcal{I}$ based on a frame contained in $\mathcal{C}$).

A formula φ is true in an interpretation $\mathcal{I}$ (symbolically $\mathcal{I} \models \varphi$) if φ is true in all worlds of $\mathcal{I}$.

A formula φ is valid in a frame $\mathcal{F}$ or $\mathcal{F}$-valid (symb. $\mathcal{F} \models \varphi$) if φ is true in all interpretations based on $\mathcal{F}$.

A formula φ is valid in a class of frames $\mathcal{C}$ or $\mathcal{C}$-valid (symb. $\mathcal{C} \models \varphi$) if $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathcal{C}$.

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K denotes the class of all frames – named after Saul Kripke, who invented this semantics.

Some validities in **K**:

1. $\varphi \vee \neg\varphi$
2. $\Box(\varphi \vee \neg\varphi)$
3. $\Box\varphi$, if φ is a classical tautology
4. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (axiom schema K)

Moreover, it holds:

If φ is **K**-valid, then $\Box\varphi$ is **K**-valid

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If φ is **K**-valid, then $\Box\varphi$ is **K**-valid

K denotes the class of all frames – named after Saul Kripke, who invented this semantics.

Some validities in **K**:

1. $\varphi \vee \neg\varphi$
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Validity: some examples

Theorem

*K is **K**-valid.* $K = \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Proof.

Let $\mathcal{I}$ be an interpretation and let w be a world in $\mathcal{I}$.
Assume $\mathcal{I}, w \models \Box(\varphi \to \psi)$, i.e., in all worlds u with wRu, if φ is true then also ψ is. (Otherwise K is true in w anyway.)
If $\Box\varphi$ is false in w, then $(\Box\varphi \to \Box\psi)$ is true and K is true in w.
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Non-validity: example

Proposition

$\Diamond\top$ *is not* **K***-valid.*

Proof.

A counterexample is the following interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ with:

$$W := \{w\},$$
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$$\pi_w(a) := T \quad (a \in \Sigma).$$

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Non-validity: another example

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$\Box\varphi \to \Box\Box\varphi$ *is not* **K***-valid.*

Proof.

A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{u,v,w\}, \{(u,v),(v,w)\}, \pi \rangle$$

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Accessibility and axiom schemata

Let us consider the following axiom schemata:

T:	$\Box\varphi \rightarrow \varphi$	(knowledge axiom)
4:	$\Box\varphi \rightarrow \Box\Box\varphi$	(positive introspection)
5:	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$	(or $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: negative introspection)
B:	$\varphi \rightarrow \Box\Diamond\varphi$	
D:	$\Box\varphi \rightarrow \Diamond\varphi$	(or $\Box\varphi \rightarrow \neg\Box\neg\varphi$: disbelief in the negation)

... and the following classes of frames, for which the accessibility relation is restricted as follows:

- T: reflexive (*wRw* for each world *w*)
- 4: transitive (*wRu* and *uRv* implies *wRv*)
- 5: euclidian (*wRu* and *wRv* implies *uRv*)
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Correspondence between accessibility relations and axiom schemata (1)

Theorem

Axiom schema T $(4, 5, B, D)$ is **T**- *valid (***4-, 5-, B-***, or* **D***-valid, respectively).*

Proof.

For T and **T**: Let $\mathcal{F}$ be a frame from class **T**. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let w be an arbitrary world in $\mathcal{I}$.
If $\Box\varphi$ is not true in world w, then axiom T is true in w.
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Motivation
Syntax
Semantics
Possible worlds
Kripke semantics
Basic notions
Relational properties vs. axioms
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Construct an interpretation $\mathcal{I}$ such that $\mathcal{I}, w \not\models a$ and $\mathcal{I}, v \models a$ for all v such that wRv.

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Different Logics

Name	Property	Axiom schema
K	–	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
T	reflexivity	$\Box\varphi \rightarrow \varphi$
4	transitivity	$\Box\varphi \rightarrow \Box\Box\varphi$
5	euclidicity	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$
B	symmetry	$\varphi \rightarrow \Box\Diamond\varphi$
D	seriality	$\Box\varphi \rightarrow \Diamond\varphi$

Some basic modal logics:

$$\begin{array}{rcl} K & & \\ KT4 & = & S4 \\ KT5 & = & S5 \\ \vdots & & \end{array}$$

Different modal logics

logics	$\Box$	$\Diamond = \neg\Box\neg$	K	T	4	5	B	D
alethic	necessarily	possibly	Y	Y	Y	Y	Y	Y
epistemic	known	possible	Y	Y	Y	Y	Y	Y
doxastic	believed	possible	Y	N	Y	Y	N	Y
deontic	obligatory	permitted	Y	N	Y?	Y?	N	Y
temporal	always (in the future)	sometimes (...)	Y	Y/N	Y	N	N	N/Y

Analytic Tableaux

Proof methods

- How can we show that a formula is $\mathcal{C}$-valid?
- In order to show that a formula is not $\mathcal{C}$-valid, one can construct a counterexample (= an interpretation that falsifies it).
- When trying out all ways of generating a counterexample without success, this counts as a proof of validity.

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Tableaux method

A tableau is a tree with nodes marked as follows:

- $w \models \varphi$,
- $w \not\models \varphi$, and
- wRv.

A branch that contains nodes marked with $w \models \varphi$ and $w \not\models \varphi$ is closed. All other branches are open. If all branches are closed, the tableau is called closed.

A tableau is constructed by using the tableau rules.

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Tableau rules for propositional logic

$$\frac{w \models \varphi \vee \psi}{w \models \varphi \mid w \models \psi}$$

$$\frac{w \not\models \varphi \vee \psi}{\begin{array}{c} w \not\models \varphi \\ w \not\models \psi \end{array}}$$

$$\frac{w \models \neg\varphi}{w \not\models \varphi}$$

$$\frac{w \models \varphi \wedge \psi}{\begin{array}{c} w \models \varphi \\ w \models \psi \end{array}}$$

$$\frac{w \not\models \varphi \wedge \psi}{w \not\models \varphi \mid w \not\models \psi}$$

$$\frac{w \not\models \neg\varphi}{w \models \varphi}$$

$$\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi \mid w \models \psi}$$

$$\frac{w \not\models \varphi \rightarrow \psi}{\begin{array}{c} w \models \varphi \\ w \not\models \psi \end{array}}$$

Additional tableau rules for modal logic **K**

$$\frac{w \models \Box\varphi}{v \models \varphi}$$ if wRv is on the branch already

$$\frac{w \not\models \Box\varphi}{\begin{array}{c} wRv \\ v \not\models \varphi \end{array}}$$ for new v

$$\frac{w \models \Diamond\varphi}{\begin{array}{c} wRv \\ v \models \varphi \end{array}}$$ for new v

$$\frac{w \not\models \Diamond\varphi}{v \not\models \varphi}$$ if wRv is on the branch already

Properties of **K** tableaux

Proposition

If a K-tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness)

If a K-tableau with root $w \not\models \varphi$ is closed, then φ is **K***-valid.*

Theorem (Completeness)

If φ is **K***-valid, then there is a closed tableau with root $w \not\models \varphi$.*

Proposition (Termination)

There are strategies for constructing **K***-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.*

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Tableau rules for other modal logics

Proofs within more restricted classes of frames allow the use of further tableau rules.

- For reflexive (**T**) frames we may extend any branch with wRw.
- For transitive (**4**) frames we have the following additional rule:
  - If wRv and vRu are in a branch, wRu may be added to the branch.
- For serial (**D**) frames we have the following rule:
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- Similar rules for other properties...

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Complextity of simple modal logics

How hard is it to check whether a modal logic formula is satisfiable or valid?

The answer depends in fact on the considered class of frames!

For example, one can show that each formula φ that is satisfiable in some S5-frame is satisfiable in an S5-frame with $|W| \leq |\varphi|$.

Proposition

Checking whether a modal formula is satisfiable in some S5-model is NP-complete (and hence checking S5-validity is coNP-complete).

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Testing logical consequence with tableaux

Let X be a class of frames.
Let Θ denote a (finite) set of formulae.
Define a consequence relation $\Theta \models_X \varphi$ as follows:
For each interpretation $\mathcal{I}$ based on a frame in X, if $\mathcal{I} \models \psi$ for each $\psi \in \Theta$, then $\mathcal{I} \models \varphi$.

- How can we check whether $\Theta \models \varphi$?
- Can we apply some kind of deduction theorem as in propositional logic:

$$\Theta \cup \{\psi\} \models_{PL} \varphi \Rightarrow \Theta \models_{PL} \psi \to \varphi \ ?$$

- Example: $a \models_{\mathsf{K}} \Box a$ holds, but $a \to \Box a$ is not K-valid.
- There is no deduction theorem as in propositional logic, and logical consequence cannot be directly reduced to validity!

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For testing logical consequence, we can use the following tableau rule:

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Embedding in FOL

Connection between propositional modal logic and FOL?

- There are similarities between predicate logic and propositional modal logics:
  1. $\Box$ vs. $\forall$
  2. $\Diamond$ vs. $\exists$
  3. possible worlds vs. objects of the universe
- In fact, many propositional modal logics can be embedded in the predicate logic.

$\Rightarrow$ Modal logics can be understood as a sublanguage of FOL.

Embedding modal logics into FOL (1)

1. $\tau(p, x) = p(x)$ for propositional variables p
2. $\tau(\neg\varphi, x) = \neg\tau(\varphi, x)$
3. $\tau(\varphi \vee \psi, x) = \tau(\varphi, x) \vee \tau(\psi, x)$
4. $\tau(\varphi \wedge \psi, x) = \tau(\varphi, x) \wedge \tau(\psi, x)$
5. $\tau(\Box\varphi, x) = \forall y (R(x, y) \rightarrow \tau(\varphi, y))$ for some new y
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Embedding modal logics into FOL (2)

Theorem

φ is K-valid if and only if $\forall x\, \tau(\varphi, x)$ is valid in FOL.

Theorem

φ is T-valid if and only if in FOL the logical consequence $\{\forall x R(x,x)\} \models \forall x\, \tau(\varphi, x)$ holds.

Example

$\Box p \wedge \Diamond(p \rightarrow q) \rightarrow \Diamond q$ is K-valid, because

$$\forall x (\forall x'(R(x,x') \rightarrow p(x')) \wedge \exists x'(R(x,x') \wedge (p(x') \rightarrow q(x'))) \\ \rightarrow \exists x'(R(x,x') \wedge q(x')))$$

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Motivation
Syntax
Semantics
Different Logics
Analytic Tableaux
Embedding in FOL
Outlook & literature

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Outlook & literature

Outlook

We only looked at some basic propositional modal logics. There are also:

- modal first order logics (with quantification $\forall$ and $\exists$ and predicates)
- multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents
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Literature I

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