Principles of Knowledge Representation and Reasoning

Complexity Theory

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1 Motivation

Why complexity theory?

- Complexity theory can answer questions on how easy or hard a problem is
- Gives hints on what algorithms could be appropriate, e.g.:
  - algorithms for polynomial-time problems are usually easy to design
  - for NP-complete problems, backtracking and local search work well
- Gives hints on what type of algorithm will (most probably) not work
  - for problems that are believed to be harder than NP-complete ones, simple backtracking will not work
- Gives hint on what sub-problems might be interesting

2 Reminder: Basic Notions

- Algorithms and Turing machines
- Problems, solutions, and complexity
- Complexity classes P and NP
- Upper and lower bounds
- Polynomial reductions
- NP-completeness

Algorithms and Turing machines

- We use Turing machines as formal models of algorithms
- This is justified, because:
  - we assume that Turing machines can compute all computable functions
  - the resource requirements (in term of time and memory) of a Turing machine are only polynomially worse than other models
- The regular type of Turing machine is the deterministic one: DTM (or simply TM)
- Often, however, we use the notion of nondeterministic TMs: NDTM

Problems, solutions, and complexity

- A problem is a set of pairs (I, A) of strings in $\{0,1\}^*$.
  I: instance; A: answer
  If all answers $A \in \{0,1\}$: decision problem
- A decision problem is the same as a formal language: the set of strings formed by the instances with answer 1
- An algorithm decides (or solves) a problem if it computes the right answer for all instances.
- Complexity of an algorithm: function

$$T : \mathbb{N} \to \mathbb{N},$$

  measuring the number of basic steps (or memory requirement) the algorithm needs to compute an answer depending on the size of the instance
- Complexity of a problem: complexity of the most efficient algorithm that solves this problem.

Complexity classes P and NP

Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: P
  - Problems in P are assumed to be efficiently solvable (although this might not be true if the exponent is very large)
  - In practice, this notion appears to be more often reasonable than not
- The class of problems decidable on non-deterministic Turing machines in polynomial time: NP
- More classes are definable using other resource bounds on time and memory

Upper and lower bounds

- Upper bounds (membership in a class) are usually easy to prove:
  - provide an algorithm
  - show that the resource bounds are respected
- Lower bounds (hardness for a class) are usually difficult to show:
  - the technical tool here is the polynomial reduction (or any other appropriate reduction)
  - show that some hard problem can be reduced to the problem at hand

Polynomial reduction

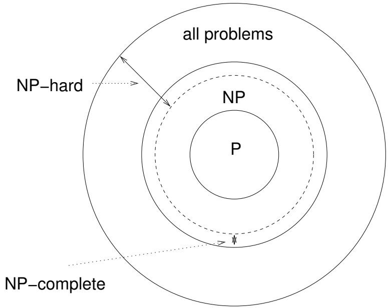
- Given languages L_1 and L_2, L_1 can be polynomially reduced to L_2, written $L_1 \leq_p L_2$, if there exists a polynomially computable function f such that

$$x \in L_1 \iff f(x) \in L_2.$$

  Rationale: it cannot be harder to decide L_1 than L_2
- L is hard for a class C (C-hard) if all languages of this class can be reduced to L.
- L is complete for C (C-complete) if L is C-hard and $L \in C$.

NP-complete problems

- A problem is NP-complete iff it is NP-hard and in NP.
- Example: SAT (the satisfiability problem for propositional logic) is NP-complete (Cook/Karp)
  - Membership is obvious, hardness follows because computations on a NDTM correspond to satisfying truth-assignments of certain formulae



3 Beyond NP

- The class co-NP
- The class PSPACE
- Other classes

The complexity class co-NP

- Note that there is some asymmetry in the definition of NP:
  - It is clear that we can decide SAT by using a NDTM with polynomially bounded computation
  - There exists an accepting computation of polynomial length iff the formula is satisfiable
  - What if we want to solve UNSAT, the complementary problem?
  - It seems necessary to check all possible truth-assignments!
- Define co-C $= \{\Sigma^* \setminus L \colon L \subseteq \Sigma^* \text{ and } L \in C\}$ (Σ ranges over alphabets)
- co-NP = $\{\Sigma^* \setminus L \colon L \subseteq \Sigma^* \text{ and } L \in \text{NP}\}$
- Examples: UNSAT, TAUT $\in$ co-NP!
- *Note:* P is closed under complement, in particular,

$$\text{P} \subseteq \text{NP} \cap \text{co-NP}$$

PSPACE

There are problems even more difficult than NP and co-NP…

Definition ((N)PSPACE)

PSPACE (NPSPACE) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only polynomially many tape cells.

Some facts about PSPACE:

- PSPACE is closed under complements (… as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- NP$\subseteq$PSPACE (because in polynomial time one can "visit" only polynomial space, i.e., NP$\subseteq$NPSPACE)
- It is unknown whether NP$\neq$PSPACE, but it is believed that

Definition (PSPACE-completeness)

A decision problem (or language) is PSPACE-complete if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, PSPACE-complete problems are the “hardest” problems in PSPACE (similar to NP-completeness). They appear to be “harder” than NP-complete problems from a practical point of view.

An example for a PSPACE-complete problem is the NDFA equivalence problem:

Instance: Two non-deterministic finite state automata A_1 and A_2.

Question: Are the languages accepted by A_1 and A_2 identical?

Other complexity classes ...

- There are complexity classes above PSPACE (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME ...)
- There are (infinitely many) classes between NP and PSPACE (the polynomial hierarchy defined by oracle machines)
- There are (infinitely many) classes inside P (circuit classes with different depths)
- ... and for most of the classes we do not know whether the containment relationships are strict

4 Oracle TMs and the Polynomial Hierarchy

- Oracle Turing machines
- Turing reduction
- Complexity classes based on OTMs
- QBF

Oracle Turing machines

- An Oracle Turing machine ((N)OTM) is a Turing machine (DTM, NDTM) with the possibility to query an oracle (i. e., a different Turing machine without resource restrictions) whether it accepts or rejects a given string.
- Computation by the oracle does not cost anything!
- Formalization:
  - a tape onto which strings for the oracle are written,
  - a yes/no answer from the oracle depending on whether it accepts or rejects the input string.
- Usage of OTMs answers what-if questions: What if we could solve the oracle-problem efficiently?

Turing reductions

- OTMs allow us to define a more general type of reduction
- Idea: The "classical" reduction can be seen as calling a subroutine once.
- L_1 is Turing-reducible to L_2, symbolically $L_1 \leq_T L_2$, if there exists a poly-time OTM that decides L_1 by using an oracle for L_2.
- Polynomial reducibility implies Turing reducibility, but not vice versa!
- NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent!
- Turing reducibility can also be applied to general search problems!

Complexity classes based on Oracle TMs

1. P^{NP} = decision problems solved by poly-time DTMs with an oracle for a decision problem in NP.
2. $\mathrm{NP}^{\mathrm{NP}}$ = decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
3. $\text{co-NP}^{\mathrm{NP}}$ = complements of decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
4. $\mathrm{NP}^{\mathrm{NP}^{\mathrm{NP}}}$ = ...

... and so on

Example

Consider the Minimum Equivalent Expression (MEE) problem:

Instance: A well-formed Boolean formula φ using the standard connectives (not $\leftrightarrow$) and a non-negative integer k.

Question: Is there a well-formed Boolean formula φ' that contains k or fewer literal occurrences and that is logically equivalent to φ?

- This problem is NP-hard (wrt. to Turing reductions).
- It does not appear to be NP-complete.
- We could guess a formula and then use a SAT-oracle …
- MEE $\in$ $\mathrm{NP}^{\mathrm{NP}}$.

The polynomial hierarchy

The complexity classes based on OTMs form an infinite hierarchy.

The polynomial hierarchy PH

$$
\begin{array}{rclrclrcl}
\Sigma_0^p & = & \mathrm{P} & \Pi_0^p & = & \mathrm{P} & \Delta_0^p & = & \mathrm{P} \\
\Sigma_{i+1}^p & = & \mathrm{NP}^{\Sigma_i^p} & \Pi_{i+1}^p & = & \text{co-}\Sigma_{i+1}^p & \Delta_{i+1}^p & = & \mathrm{P}^{\Sigma_i^p}
\end{array}
$$

- PH = $\bigcup_{i \geq 0}(\Sigma_i^p \cup \Pi_i^p \cup \Delta_i^p) \subseteq$PSPACE
- NP = Σ_1^p
- co-NP = Π_1^p

Quantified Boolean formulae: definition

- If φ is a propositional formula, P is the set of Boolean variables used in φ and σ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma\varphi$ is a QBF.
- A formula $\exists x \varphi$ is true if and only if $\varphi[x/\top] \vee \varphi[x/\bot]$ is true (equivalently, $\varphi[x/\top]$ is true or $\varphi[x/\bot]$ is true).
- A formula $\forall x \varphi$ is true if and only if $\varphi[x/\top] \wedge \varphi[x/\bot]$ is true (equivalently, $\varphi[x/\top]$ is true and $\varphi[x/\bot]$ is true).
- This definition directly leads to an AND/OR tree traversal algorithm for evaluating QBF.

Quantified Boolean formulae: definition

The evaluation problem of QBF generalizes both the satisfiability and validity/tautology problems of propositional logic.
The latter are NP-complete and co-NP-complete, resp., whereas the former is PSPACE-complete.

Example

The formulae $\forall x \exists y (x \leftrightarrow y)$ and $\exists x \exists y (x \land y)$ are true.

Example

The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \lor y)$ are false.

The Polynomial Hierarchy: connection to QBF

Truth of QBFs with prefix $\overbrace{\forall\exists\forall\ldots}^{i}$ is Π_i^p-complete.

Truth of QBFs with prefix $\overbrace{\exists\forall\exists\ldots}^{i}$ is Σ_i^p-complete.

Special cases corresponding to SAT and TAUT:

- The truth of QBFs with prefix $\exists x_1^1 \ldots x_n^1$ is NP$= \Sigma_1^p$-complete.
- The truth of QBFs with prefix $\forall x_1^1 \ldots x_n^1$ is co-NP$= \Pi_1^p$-complete.

Literature

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