Principles of Knowledge Representation and Reasoning

Predicate logic

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1 Motivation

Why first-order logic (FOL)?

- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - All CS students know formal logic
  - Peter is a CS student
  - Therefore, Peter knows formal logic
  - Not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.

2 Syntax

Syntax

- variable symbols: $x, y, z, \ldots$
- n-ary function symbols: $f, g, \ldots$
- constant symbols: $a, b, c, \ldots$
- n-ary predicate symbols: $P, Q, \ldots$
- logical symbols: $\forall$, $\exists$, $=$, $\neg$, $\wedge$, $\ldots$

Terms	t	$\longrightarrow x$	variable
		$\mid\ f(t_1, \ldots, t_n)$	function application
		$\mid\ a$	constant
Formulae	φ	$\longrightarrow P(t_1, \ldots, t_n)$	atomic formulae
		$\mid\ t = t'$	identity formulae
		$\mid\ \ldots$	propositional connectives
		$\mid\ \forall x \varphi'$	universal quantification
		$\mid\ \exists x \varphi'$	existential quantification

ground term, etc.: term, etc. without variable occurrences

3 Semantics

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- Interpretations
- Variable Maps
- Definition of Truth
- Terminology
- Free and Bound Variables
- Open and Closed Formulae

Semantics: idea

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- In FOL, the universe of discourse consists of objects, functions over these objects, and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $\mathcal{I}(x)$ we write $x^{\mathcal{I}}$.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt all these universes.

Formal semantics: interpretations

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Interpretations: $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{D}$ being an arbitrary non-empty set and $\cdot^{\mathcal{I}}$ being a function which maps

- n-ary function symbols f to n-ary functions $f^{\mathcal{I}} \in [\mathcal{D}^n \to \mathcal{D}]$,
- constant symbols a to objects $a^{\mathcal{I}} \in \mathcal{D}$, and
- n-ary predicates P to n-ary relations $P^{\mathcal{I}} \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^{\mathcal{I}} \quad = \quad f^{\mathcal{I}}(t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}}) \ (\in \mathcal{D})$$

Truth of ground atoms:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

Examples

$$
\begin{aligned}
\mathcal{D} &= \{d_1,\dots,d_n\}, n \geq 2 \\
\mathrm{a}^{\mathcal{I}} &= d_1 \\
\mathrm{b}^{\mathcal{I}} &= d_2 \\
\mathrm{eye}^{\mathcal{I}} &= \{d_1\} \\
\mathrm{red}^{\mathcal{I}} &= \mathcal{D} \\
\mathcal{I} &\models \mathrm{red(b)} \\
\mathcal{I} &\not\models \mathrm{eye(b)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D} &= \{1,2,3,\dots\} \\
1^{\mathcal{I}} &= 1 \\
2^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathrm{even}^{\mathcal{I}} &= \{2,4,6,\dots\} \\
\mathrm{succ}^{\mathcal{I}} &= \{(1 \mapsto 2),(2 \mapsto 3),\dots\} \\
\mathcal{I} &\not\models \mathrm{even}(3) \\
\mathcal{I} &\models \mathrm{even}(\mathrm{succ}(3))
\end{aligned}
$$

Formal semantics: variable maps

V is the set of variables. Function $\alpha: V \to \mathcal{D}$ is a variable map.
Notation: $\alpha[x/d]$ is identical to α except for x where $\alpha[x/d](x) = d$.
Interpretation of terms under $\mathcal{I},\alpha$:

$$
\begin{aligned}
x^{\mathcal{I},\alpha} &= \alpha(x) \\
a^{\mathcal{I},\alpha} &= a^{\mathcal{I}} \\
(f(t_1,\dots,t_n))^{\mathcal{I},\alpha} &= f^{\mathcal{I}}(t_1{}^{\mathcal{I},\alpha},\dots,t_n{}^{\mathcal{I},\alpha})
\end{aligned}
$$

Truth of atomic formulae:

$$\mathcal{I},\alpha \models P(t_1,\dots,t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I},\alpha},\dots,t_n{}^{\mathcal{I},\alpha}\rangle \in P^{\mathcal{I}}$$

Example (cont'd):
$\alpha = \{x \mapsto d_1, y \mapsto d_2\}$ $\quad \mathcal{I},\alpha \models \mathrm{red}(x)$ $\quad \mathcal{I},\alpha[y/d_1] \models \mathrm{eye}(y)$

Formal semantics: truth

Truth of φ by $\mathcal{I}$ under α ($\mathcal{I},\alpha \models \varphi$) is defined as follows.

$\mathcal{I},\alpha \models P(t_1,\dots,t_n)$	iff $\langle t_1{}^{\mathcal{I},\alpha},\dots,t_n{}^{\mathcal{I},\alpha}\rangle \in P^{\mathcal{I}}$
$\mathcal{I},\alpha \models t_1 = t_2$	iff $t_1{}^{\mathcal{I},\alpha} = t_2{}^{\mathcal{I},\alpha}$
$\mathcal{I},\alpha \models \neg\varphi$	iff $\mathcal{I},\alpha \not\models \varphi$
$\mathcal{I},\alpha \models \varphi \wedge \psi$	iff $\mathcal{I},\alpha \models \varphi$ and $\mathcal{I},\alpha \models \psi$
$\mathcal{I},\alpha \models \varphi \vee \psi$	iff $\mathcal{I},\alpha \models \varphi$ or $\mathcal{I},\alpha \models \psi$
$\mathcal{I},\alpha \models \varphi \to \psi$	iff if $\mathcal{I},\alpha \models \varphi$, then $\mathcal{I},\alpha \models \psi$
$\mathcal{I},\alpha \models \varphi \leftrightarrow \psi$	iff $\mathcal{I},\alpha \models \varphi$, iff $\mathcal{I},\alpha \models \psi$
$\mathcal{I},\alpha \models \forall x\, \varphi$	iff $\mathcal{I},\alpha[x/d] \models \varphi$ for all $d \in \mathcal{D}$
$\mathcal{I},\alpha \models \exists x\, \varphi$	iff $\mathcal{I},\alpha[x/d] \models \varphi$ for some $d \in \mathcal{D}$

Examples

$$
\begin{aligned}
\Theta &= \left\{ \begin{array}{l} \mathrm{eye}(a), \mathrm{eye}(b) \\ \forall x(\mathrm{eye}(x) \to \mathrm{red}(x)) \end{array} \right\} \\
\mathcal{D} &= \{d_1,\dots,d_n\},\ n > 1 \\
\mathrm{a}^{\mathcal{I}} &= d_1 \\
\mathrm{b}^{\mathcal{I}} &= d_1 \\
\mathrm{eye}^{\mathcal{I}} &= \{d_1\} \\
\mathrm{red}^{\mathcal{I}} &= \mathcal{D} \\
\alpha &= \{(x \mapsto d_1),(y \mapsto d_2)\}
\end{aligned}
$$

Questions:

$\mathcal{I},\alpha \models \mathrm{eye}(b) \vee \neg\mathrm{eye}(b)$? Yes

$\mathcal{I},\alpha \models \mathrm{eye}(x) \to \mathrm{eye}(x) \vee \mathrm{eye}(y)$? Yes

$\mathcal{I},\alpha \models \mathrm{eye}(x) \to \mathrm{eye}(y)$? No

$\mathcal{I},\alpha \models \mathrm{eye}(a) \wedge \mathrm{eye}(b)$? Yes

$\mathcal{I},\alpha \models \forall x(\mathrm{eye}(x) \to \mathrm{red}(x))$? Yes

$\mathcal{I},\alpha \models \Theta$? Yes

Terminology

$\mathcal{I}, \alpha$ is a model of φ iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be satisfiable, unsatisfiable, falsifiable, valid, ...

Two formulae φ and ψ are logically equivalent (symb.: $\varphi \equiv \psi$) iff for all $\mathcal{I}, \alpha$:

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

Note: $\mathrm{P(x)} \not\equiv \mathrm{P(y)}$!

Logical implication is also analogous to propositional logic:

$$\Theta \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

Free and bound variables

Variables can be free or bound (by a quantifier) in a formula:

$$\begin{aligned}
\mathrm{free}(x) &= \{x\} \\
\mathrm{free}(f(t_1, \dots, t_n)) &= \mathrm{free}(t_1) \cup \dots \cup \mathrm{free}(t_n) \\
\mathrm{free}(t_1 = t_2) &= \mathrm{free}(t_1) \cup \mathrm{free}(t_2) \\
\mathrm{free}(P(t_1, \dots, t_n)) &= \mathrm{free}(t_1) \cup \dots \cup \mathrm{free}(t_n) \\
\mathrm{free}(\neg\varphi) &= \mathrm{free}(\varphi) \\
\mathrm{free}(\varphi * \psi) &= \mathrm{free}(\varphi) \cup \mathrm{free}(\psi), \text{ for } * = \vee, \wedge, \rightarrow, \leftrightarrow \\
\mathrm{free}(\Xi x \varphi) &= \mathrm{free}(\varphi) \setminus \{x\}, \text{ for } \Xi = \forall, \exists
\end{aligned}$$

Example: $\forall \mathrm{x}\, (\mathrm{R}(\boxed{\mathrm{y}}, \boxed{\mathrm{z}}) \wedge \exists \mathrm{y}\, (\neg \mathrm{P(y,x)} \vee \mathrm{R}(\mathrm{y}, \boxed{\mathrm{z}})))$
Framed occurrences are free, all others are bound.

Open & closed formulae

- Formulae without free variables are called closed formulae or sentences. Formulae with free variables are called open formulae.
- Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of $\forall$ and $\exists$).
- Note that logical equivalence, satisfiability, and entailment are independent from variable maps if we consider only closed formulae.
- For closed formulae, we omit α in connection with $\models$:

$$\mathcal{I} \models \varphi.$$

Important theorems

Theorem (Compactness)

Let $\Phi \cup \{\psi\}$ be a set of closed formulae.

(a) *$\Phi \models \psi$ iff there exists a finite subset $\Phi' \subseteq \Phi$ s. t. $\Phi' \models \psi$.*

(b) *Φ is satisfiable iff each finite subset $\Phi' \subseteq \Phi$ is satisfiable.*

Theorem (Löwenheim-Skolem)

Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.

4 Literature

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Literature

Literature

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