# Principles of Knowledge Representation and Reasoning

## Predicate logic

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# 1 Motivation

# Why first-order logic (FOL)?



- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - All CS students know formal logic
  - Peter is a CS student
  - Therefore, Peter knows formal logic
  - Not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.

# 2 Syntax

# Syntax

- variable symbols: $x, y, z, \ldots$
- $n$-ary function symbols: $f, g, \ldots$
- constant symbols: $a, b, c, \ldots$
- $n$-ary predicate symbols: $P, Q, \ldots$
- logical symbols: $\forall$, $\exists$, $=$, $\neg$, $\wedge$, $\ldots$

| | | | | |
|---|---|---|---|---|
| Terms | $t$ | $\longrightarrow$ | $x$ | variable |
| | | $\mid$ | $f(t_1, \ldots, t_n)$ | function application |
| | | $\mid$ | $a$ | constant |
| Formulae | $\varphi$ | $\longrightarrow$ | $P(t_1, \ldots, t_n)$ | atomic formulae |
| | | $\mid$ | $t = t'$ | identity formulae |
| | | $\mid$ | $\ldots$ | propositional connectives |
| | | $\mid$ | $\forall x \varphi'$ | universal quantification |
| | | $\mid$ | $\exists x \varphi'$ | existential quantification |

ground term, etc.: term, etc. without variable occurrences

# 3 Semantics

- Interpretations
- Variable Maps
- Definition of Truth
- Terminology
- Free and Bound Variables
- Open and Closed Formulae

# Semantics: idea

- In FOL, the universe of discourse consists of objects, functions over these objects, and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $\mathcal{I}(x)$ we write $x^{\mathcal{I}}$.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt all these universes.

# Formal semantics: interpretations

Interpretations: $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{D}$ being an arbitrary non-empty set and $\cdot^{\mathcal{I}}$ being a function which maps

- $n$-ary function symbols $f$ to $n$-ary functions $f^{\mathcal{I}} \in [\mathcal{D}^n \rightarrow \mathcal{D}]$,
- constant symbols $a$ to objects $a^{\mathcal{I}} \in \mathcal{D}$, and
- $n$-ary predicates $P$ to $n$-ary relations $P^{\mathcal{I}} \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^{\mathcal{I}} \quad = \quad f^{\mathcal{I}}(t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}}) \; (\in \mathcal{D})$$

Truth of ground atoms:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

# Examples

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$$
\begin{aligned}
\mathcal{D} &= \{d_1, \ldots, d_n\}, n \geq 2 \\
\mathrm{a}^{\mathcal{I}} &= d_1 \\
\mathrm{b}^{\mathcal{I}} &= d_2 \\
\mathrm{eye}^{\mathcal{I}} &= \{d_1\} \\
\mathrm{red}^{\mathcal{I}} &= \mathcal{D} \\
\mathcal{I} &\models \mathrm{red}(\mathrm{b}) \\
\mathcal{I} &\not\models \mathrm{eye}(\mathrm{b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D} &= \{1, 2, 3, \ldots\} \\
1^{\mathcal{I}} &= 1 \\
2^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathrm{even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\
\mathrm{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\
\mathcal{I} &\not\models \mathrm{even}(3) \\
\mathcal{I} &\models \mathrm{even}(\mathrm{succ}(3))
\end{aligned}
$$

# Formal semantics: variable maps

$V$ is the set of variables. Function $\alpha\colon V \to \mathcal{D}$ is a variable map.

Notation: $\alpha[x/d]$ is identical to $\alpha$ except for $x$ where $\alpha[x/d](x) = d$.

Interpretation of terms under $\mathcal{I}, \alpha$:

$$\begin{aligned} x^{\mathcal{I},\alpha} &= \alpha(x) \\ a^{\mathcal{I},\alpha} &= a^{\mathcal{I}} \\ (f(t_1,\dots,t_n))^{\mathcal{I},\alpha} &= f^{\mathcal{I}}({t_1}^{\mathcal{I},\alpha},\dots,{t_n}^{\mathcal{I},\alpha}) \end{aligned}$$

Truth of atomic formulae:

$$\mathcal{I},\alpha \models P(t_1,\dots,t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I},\alpha},\dots,{t_n}^{\mathcal{I},\alpha}\rangle \in P^{\mathcal{I}}$$

Example (cont'd):

$\alpha = \{x \mapsto d_1, y \mapsto d_2\}$ $\qquad \mathcal{I},\alpha \models \mathrm{red}(x)$ $\qquad \mathcal{I},\alpha[y/d_1] \models \mathrm{eye}(y)$

# Formal semantics: truth

Truth of $\varphi$ by $\mathcal{I}$ under $\alpha$ ($\mathcal{I}, \alpha \models \varphi$) is defined as follows.

$$
\begin{array}{ll}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff } \langle t_1{}^{\mathcal{I},\alpha}, \ldots, t_n{}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models t_1 = t_2 & \text{iff } t_1{}^{\mathcal{I},\alpha} = t_2{}^{\mathcal{I},\alpha} \\
\mathcal{I}, \alpha \models \neg\varphi & \text{iff } \mathcal{I}, \alpha \not\models \varphi \\
\mathcal{I}, \alpha \models \varphi \wedge \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \vee \psi & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \rightarrow \psi & \text{iff if } \mathcal{I}, \alpha \models \varphi, \text{then } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \varphi \leftrightarrow \psi & \text{iff } \mathcal{I}, \alpha \models \varphi, \text{ iff } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x\, \varphi & \text{iff } \mathcal{I}, \alpha[x/d] \models \varphi \text{ for all } d \in \mathcal{D} \\
\mathcal{I}, \alpha \models \exists x\, \varphi & \text{iff } \mathcal{I}, \alpha[x/d] \models \varphi \text{ for some } d \in \mathcal{D}
\end{array}
$$

# Examples

$$\Theta = \left\{ \begin{array}{l} \mathrm{eye}(a), \mathrm{eye}(b) \\ \forall x(\mathrm{eye}(x) \rightarrow \mathrm{red}(x)) \end{array} \right\}$$

$$\mathcal{D} = \{d_1, \ldots, d_n\},\ n > 1$$

$$\mathrm{a}^{\mathcal{I}} = d_1$$

$$\mathrm{b}^{\mathcal{I}} = d_1$$

$$\mathrm{eye}^{\mathcal{I}} = \{d_1\}$$

$$\mathrm{red}^{\mathcal{I}} = \mathcal{D}$$

$$\alpha = \{(x \mapsto d_1), (y \mapsto d_2)\}$$

Questions:

$\mathcal{I}, \alpha \models \mathrm{eye}(b) \vee \neg\mathrm{eye}(b)$? Yes

$\mathcal{I}, \alpha \models \mathrm{eye}(x) \rightarrow \mathrm{eye}(x) \vee \mathrm{eye}(y)$? Yes

$\mathcal{I}, \alpha \models \mathrm{eye}(x) \rightarrow \mathrm{eye}(y)$? No

$\mathcal{I}, \alpha \models \mathrm{eye}(a) \wedge \mathrm{eye}(b)$? Yes

$\mathcal{I}, \alpha \models \forall x(\mathrm{eye}(x) \rightarrow \mathrm{red}(x))$? Yes

$\mathcal{I}, \alpha \models \Theta$? Yes

# Terminology

$\mathcal{I}, \alpha$ is a model of $\varphi$ iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be satisfiable, unsatisfiable, falsifiable, valid, ...

Two formulae $\varphi$ and $\psi$ are logically equivalent (symb.: $\varphi \equiv \psi$) iff for all $\mathcal{I}, \alpha$:

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

Note: $\mathrm{P(x)} \not\equiv \mathrm{P(y)}$!

Logical implication is also analogous to propositional logic:

$$\Theta \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

# Free and bound variables



Variables can be free or bound (by a quantifier) in a formula:

$$
\begin{aligned}
\mathrm{free}(x) &= \{x\} \\
\mathrm{free}(f(t_1,\ldots,t_n)) &= \mathrm{free}(t_1)\cup\cdots\cup\mathrm{free}(t_n) \\
\mathrm{free}(t_1 = t_2) &= \mathrm{free}(t_1)\cup\mathrm{free}(t_2) \\
\mathrm{free}(P(t_1,\ldots,t_n)) &= \mathrm{free}(t_1)\cup\cdots\cup\mathrm{free}(t_n) \\
\mathrm{free}(\neg\varphi) &= \mathrm{free}(\varphi) \\
\mathrm{free}(\varphi * \psi) &= \mathrm{free}(\varphi)\cup\mathrm{free}(\psi),\ \text{for } * = \vee,\wedge,\rightarrow,\leftrightarrow \\
\mathrm{free}(\Xi x\varphi) &= \mathrm{free}(\varphi)\setminus\{x\},\ \text{for } \Xi = \forall,\exists
\end{aligned}
$$

Example: $\forall x\ (R(\boxed{y},\boxed{z}) \wedge \exists y\ (\neg P(y,x) \vee R(y,\boxed{z})))$

Framed occurrences are free, all others are bound.

# Open & closed formulae

- Formulae without free variables are called closed formulae or sentences. Formulae with free variables are called open formulae.
- Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of $\forall$ and $\exists$).
- Note that logical equivalence, satisfiability, and entailment are independent from variable maps if we consider only closed formulae.
- For closed formulae, we omit $\alpha$ in connection with $\models$:

$$\mathcal{I} \models \varphi.$$

# Important theorems

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## Theorem (Compactness)

*Let $\Phi \cup \{\psi\}$ be a set of closed formulae.*

(a) $\Phi \models \psi$ *iff there exists a finite subset* $\Phi' \subseteq \Phi$ *s. t.* $\Phi' \models \psi$.

(b) $\Phi$ *is satisfiable iff each finite subset* $\Phi' \subseteq \Phi$ *is satisfiable.*

## Theorem (Löwenheim-Skolem)

*Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.*

# 4 Literature

# Literature



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