Principles of Knowledge Representation and Reasoning Propositional Logic

Albert-Ludwigs-Universität Freiburg

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Why Logic?

Propositio-

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Problems and

Resolution

Decision

Bernhard Nebel, Stefan Wölfl, and Julien Hué October 24, 2012

Why logic?

knowledge.

- Can be used for analysis, design and specification.

■ Logic is one of the best developed systems for representing

Understanding formal logic is a prerequisite for understanding most research papers in KRR.

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The right logic...

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- Logics of different orders (1st, 2nd, ...)
- Modal logics
 - epistemic
 - temporal
 - dynamic (program)
 - multi-modal logics
 - ...
- Many-valued logics
- Nonmonotonic logics
- Intuitionistic logics
- ...

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The logical approach

- UNI FREIBURG
- Define a formal language: logical & non-logical symbols, syntax rules
- Provide language with compositional semantics
 - Fix universe of discourse
 - Specify how the non-logical symbols can be interpreted: interpretation
 - Rules how to combine interpretation of single symbols
 - Satisfying interpretation = model
 - Semantics often entails concept of logical implication/entailment
- Specify a calculus that allows to derive new formulae from old ones according to the entailment relation

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2 Propositional Logic



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Propositional logic: main ideas



- Non-logical symbols: propositional variables or atoms
 - representing propositions which cannot be decomposed
 - which can be true or false (for example: "Snow is white", "It rains")
- Logical symbols: propositional connectives such as: and (∧), or (∨), and not (¬)
- Formulae: built out of atoms and connectives
- Universe of discourse: truth values

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3 Syntax



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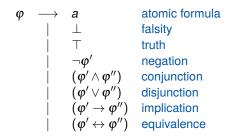
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Syntax

Countable alphabet Σ of atomic propositions: a, b, c, \dots Propositional formulae are built according to the following rule:



Parentheses can be omitted if no ambiguity arises.

Operator precedence: $\neg > \land > \lor > \rightarrow = \leftrightarrow$.

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Language and meta-language



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logic. $\phi \longrightarrow a | \dots | (\phi' \leftrightarrow \phi'')$ is a statement about how expressions in the language of propositional logic can be

 \blacksquare $(a \lor b)$ is an expression of the language of propositional

- In order to describe how expressions (in this case formulae) can be formed, we use meta-language.
- When we describe how to interpret formulae, we use meta-language expressions.

formed. It is stated using meta-language.

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Decision Problems an Semantics: idea



- \blacksquare Atomic propositions can be true (1, T) or false (0, F).
- Provided the truth values of the atoms have been fixed (truth assignment or interpretation), the truth value of a formula can be computed from the truth values of the atoms and the connectives.
- Example:

$$(a \lor b) \land c$$

is true iff c is true and, additionally, a or b is true.

Logical implication can then be defined as follows:

 \blacksquare φ is implied by a set of formulae Θ iff φ is true for all truth assignments (world states) that make all formulae in Θ true.

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Formal semantics

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An interpretation or truth assignment over Σ is a function:

$$\mathcal{I}: \Sigma \to \{T, F\}.$$

A formula ψ is true under \mathcal{I} or is satisfied by \mathcal{I} (symb. $\mathcal{I} \models \psi$):

 $\mathcal{I} \models \varphi$ if and only if $\mathcal{I} \models \varphi'$

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 $\mathcal{I} \models \varphi \leftrightarrow \varphi'$

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Given

$$\mathcal{I}: a \mapsto T, b \mapsto F, c \mapsto F, d \mapsto T,$$

Is
$$((a \lor b) \leftrightarrow (c \lor d)) \land (\neg(a \land c) \lor (c \land \neg d))$$
 true or false?

$$((\mathbf{a} \vee \mathbf{b}) \leftrightarrow (\mathbf{c} \vee \mathbf{d})) \wedge (\neg (\mathbf{a} \wedge \mathbf{c}) \vee (\mathbf{c} \wedge \neg \mathbf{d}))$$

$$((\mathbf{a} \vee \mathbf{b}) \leftrightarrow (\mathbf{c} \vee \mathbf{d})) \wedge (\neg (\mathbf{a} \wedge \mathbf{c}) \vee (\mathbf{c} \wedge \neg \mathbf{d}))$$

$$((\mathsf{a} \vee \mathsf{b}) \leftrightarrow (\mathsf{c} \vee \mathsf{d})) \wedge (\neg (\mathsf{a} \wedge \mathsf{c}) \vee (\mathsf{c} \wedge \neg \mathsf{d}))$$

$$((\mathsf{a} \vee \mathsf{b}) \leftrightarrow (\mathsf{c} \vee \mathsf{d})) \wedge (\neg (\mathsf{a} \wedge \mathsf{c}) \vee (\mathsf{c} \wedge \neg \mathsf{d}))$$

$$((\mathsf{a} \vee \mathsf{b}) \leftrightarrow (\mathsf{c} \vee \mathsf{d})) \wedge (\neg (\mathsf{a} \wedge \mathsf{c}) \vee (\mathsf{c} \wedge \neg \mathsf{d}))$$

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An interpretation ${\mathcal I}$ is a model of ${\pmb \varphi}$ iff

$$\mathcal{I} \models \varphi$$

A formula φ is

- **satisfiable** if there is an \mathcal{I} such that $\mathcal{I} \models \varphi$;
- unsatisfiable, otherwise; and
- valid if $\mathcal{I} \models \varphi$ for each \mathcal{I} (or tautology);
- falsifiable, otherwise.

Two formulae φ and ψ are logically equivalent (symb. $\varphi \equiv \psi$) if for all interpretations \mathcal{I} ,

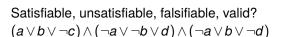
$$\mathcal{I} \models \varphi \text{ iff } \mathcal{I} \models \psi.$$

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Examples



- \rightarrow satisfiable: $a \mapsto T, b \mapsto F, d \mapsto F, \dots$
- \rightarrow falsifiable: $a \mapsto F, b \mapsto F, c \mapsto T, \dots$

$$((\neg a \to \neg b) \to (b \to a))$$

- \leadsto satisfiable: $a \mapsto T, b \mapsto T$
- valid: Consider all interpretations or argue about falsifying ones.

Equivalence?
$$\neg(a \lor b) \equiv \neg a \land \neg b$$

→ Of course, equivalent (de Morgan).

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Some obvious consequences



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Proposition

 ϕ is valid iff $\neg \phi$ is unsatisfiable and ϕ is satisfiable iff $\neg \phi$ is falsifiable.

Proposition

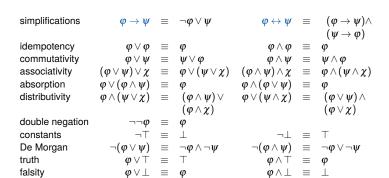
 $\varphi \equiv \psi$ iff $\varphi \leftrightarrow \psi$ is valid.

Theorem

If $\varphi \equiv \psi$ and χ' results from substituting φ by ψ in χ , then $\chi' \equiv \chi$.

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Some equivalences



 $\varphi \lor \neg \varphi \equiv \top$

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How many different formulae are there ...

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...for a given finite alphabet Σ ?

- Infinitely many: $a, a \lor a, a \land a, a \lor a \lor a, ...$
- How many different logically distinguishable (not equivalent) formulae?
 - For Σ with $n = |\Sigma|$, there are 2^n different interpretations.
 - A formula can be characterized by its set of models (if two formulae are not logically equivalent, then their sets of models differ).
 - There are $2^{(2^n)}$ different sets of interpretations.
 - There are $2^{(2^n)}$ (logical) equivalence classes of formulae.

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 $\phi \wedge \neg \phi \equiv \bot$

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Logical implication

■ Extension of the relation \models to sets Θ of formulae:

$$\mathcal{I} \models \Theta \text{ iff } \mathcal{I} \models \varphi \text{ for all } \varphi \in \Theta.$$

 ϕ is logically implied by Θ (symbolically $\Theta \models \phi$) iff ϕ is true in all models of Θ :

$$\Theta \models \varphi \text{ iff } \mathcal{I} \models \varphi \text{ for all } \mathcal{I} \text{ such that } \mathcal{I} \models \Theta$$

- Some consequences:
 - Deduction theorem: $\Theta \cup \{\phi\} \models \psi$ iff $\Theta \models \phi \rightarrow \psi$
 - **■** Contraposition: $\Theta \cup \{\phi\} \models \neg \psi$ iff $\Theta \cup \{\psi\} \models \neg \phi$
 - Contradiction: $\Theta \cup \{\phi\}$ is unsatisfiable iff $\Theta \models \neg \phi$

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Normal forms

Terminology:

- Atomic formulae a, negated atomic formulae $\neg a$, truth \top and falsity \bot are literals.
- A disjunction of literals is a clause.
- If \neg only occurs in front of an atom and there are no \rightarrow and \leftrightarrow , the formula is in negation normal form (NNF). Example: $(\neg a \lor \neg b) \land c$, but not: $\neg (a \land b) \land c$
- A conjunction of clauses is in conjunctive normal form (CNF).
 - Example: $(a \lor b) \land (\neg a \lor c)$
- The dual form (disjunction of conjunctions of literals) is in disjunctive normal form (DNF). Example: $(a \land b) \lor (\neg a \land c)$

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Negation normal form

Theorem

For each propositional formula there is a logically equivalent formula in NNF.

Proof.

First eliminate \rightarrow and \leftrightarrow by the appropriate equivalences.

Base case: Claim is true for a, $\neg a$, \top , \bot .

Inductive case: Assume claim is true for all formulae φ (up to a certain number of connectives) and call its NNF $\operatorname{nnf}(\varphi)$.

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- \blacksquare $\mathsf{nnf}(\varphi \land \psi) = (\mathsf{nnf}(\varphi) \land \mathsf{nnf}(\psi))$
- $\qquad \mathsf{nnf}(\neg(\phi \land \psi)) = (\mathsf{nnf}(\neg\phi) \lor \mathsf{nnf}(\neg\psi))$
- $\qquad \mathsf{nnf}(\neg(\varphi \lor \psi)) = (\mathsf{nnf}(\neg\varphi) \land \mathsf{nnf}(\neg\psi))$

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Conjunctive normal form

Theorem

For each propositional formula there are logically equivalent formulae in CNF and DNF, respectively.

Beweis.

The claim is true for a, $\neg a$, \top , \bot .

Let us assume it is true for all formulae φ (up to a certain number of connectives) and call its CNF $cnf(\varphi)$ (and its DNF $dnf(\varphi)$).

- $\qquad \mathsf{cnf}(\neg \varphi) = \mathsf{nnf}(\neg \, \mathsf{dnf}(\varphi)) \text{ and } \mathsf{cnf}(\varphi \wedge \psi) = \mathsf{cnf}(\varphi) \wedge \mathsf{cnf}(\psi).$
- Assume $cnf(\phi)=\bigwedge_i \chi_i$ and $cnf(\psi)=\bigwedge_j \rho_j$ with χ_i,ρ_j being clauses. Then

 $\operatorname{cnf}(\varphi\vee\psi)=\operatorname{cnf}((\bigwedge_i\chi_i)\vee(\bigwedge_j\rho_j))=\bigwedge_i\bigwedge_j(\chi_i\vee\rho_j)\quad \text{ (by distributivity)}$

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How to decide properties of formulae



How do we decide whether a formula is satisfiable, unsatisfiable, valid, or falsifiable?

Note: Satisfiability and falsifiability are NP-complete. Validity and unsatisfiability are co-NP-complete.

- A CNF formula is valid iff all clauses contain two complementary literals or T.
- \blacksquare A DNF formula is satisfiable iff one disjunct does not contain \bot or two complementary literals.
- However, transformation to CNF or DNF may take exponential time (and space!).
- One can try out all truth assignments.
- One can test systematically for satisfying truth assignments (backtracking) → Davis-Putnam-Logemann-Loveland.

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Resolution

Deciding entailment

Completeness

Horn Clauses

Resolution Strategies

- We want to decide $\Theta \models \varphi$.
- Use deduction theorem and reduce to validity:

$$\Theta \models \varphi \; \mathsf{iff} \; \bigwedge \Theta \to \varphi \; \mathsf{is} \; \mathsf{valid}.$$

- Now negate and test for unsatisfiability using DPLL.
- Different approach: Try to derive φ from Θ find a proof of φ from Θ .
- Use inference rules to derive new formulae from Θ . Continue to deduce new formulae until φ can be deduced.
- One particular calculus: resolution.

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Resolution: representation



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- We assume that all formulae are in CNF.
 - Can be generated using the described method.
 - Often formulae are already close to CNF.
 - There is a "cheap" conversion from arbitrary formulae to CNF that preserves satisfiability - which is enough as we will see.
- More convenient representation:
 - CNF formula is represented as a set.
 - Each clause is a set of literals.
- Empty clause (symbolically □) and empty set of clauses (symbolically ∅) are different!

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Strategies Horn Clauses

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Resolution: the inference rule

Let I be a literal and \overline{I} its complement.

The resolution rule

$$\frac{C_1 \dot{\cup} \{I\}, C_2 \dot{\cup} \{\bar{I}\}}{C_1 \cup C_2}$$

 $C_1 \cup C_2$ is the resolvent of the parent clauses $C_1 \cup \{l\}$ and $C_2 \cup \{\overline{I}\}$. I and \overline{I} are the resolution literals.

Example: $\{a, b, \neg c\}$ resolves with $\{a, d, c\}$ to $\{a, b, d\}$.

Note: The resolvent is not logically equivalent to the set of parent clauses!

Notation:

 $R(\Delta) = \{C \mid C \text{ is resolvent of two clauses in } \Delta\}$

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Resolution: derivations

D can be derived from Δ by resolution (symbolically $\Delta \vdash D$) if there is a sequence C_1, \ldots, C_n of clauses such that

 $C_n = D$ and

 $C_i \in R(\Delta \cup \{C_1, ..., C_{i-1}\}), \text{ for all } i \in \{1, ..., n\}.$

Define $R^*(\Delta) = \{D \mid \Delta \vdash D\}.$

Theorem (Soundness of resolution)

Let D be a clause. If $\Delta \vdash D$ then $\Delta \models D$.

Proof idea.

Show $\Delta \models D$ if $D \in R(\Delta)$ and use induction on proof length.

Let $C_1 \cup \{I\}$ and $C_2 \cup \{\overline{I}\}$ be the parent clauses of $D = C_1 \cup C_2$.

Assume $\mathcal{I} \models \Delta$, we have to show $\mathcal{I} \models D$.

Case 1: $\mathcal{I} \models I$ then $\exists m \in C_2$ s.t. $\mathcal{I} \models m$. This implies $\mathcal{I} \models D$.

Case 2: $\mathcal{I} \models \overline{l}$ similarly, $\exists m \in C_1$ s.t. $\mathcal{I} \models m$.

This means that each model \mathcal{I} of Δ also satisfies D, i.e., $\Delta \models D$.

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Resolution: completeness?

Do we have

 $\Delta \models \varphi \text{ implies } \Delta \vdash \varphi$?

Of course, could only hold for CNF. However:

 $\left\{\{a,b\},\{\neg b,c\}\right\} \models \{a,b,c\}$ $\neq \{a,b,c\}$

However, one can show that resolution is refutation-complete:

 Δ is unsatisfiable iff $\Delta \vdash \Box$.

Entailment: Reduce to unsatisfiability testing and decide by resolution.

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Resolution strategies

- Trying out all different resolutions can be very costly,
- and might not be necessary.
- There are different resolution strategies.
- Examples:
 - Input resolution ($R_l(\cdot)$): In each resolution step, one of the parent clauses must be a clause of the input set.
 - Unit resolution $(R_U(\cdot))$: In each resolution step, one of the parent clauses must be a unit clause.
 - Not all strategies are (refutation) completeness preserving. Neither input nor unit resolution is. However, there are others.

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Horn clauses & resolution



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Horn clauses: Clauses with at most one positive literal Example: $(a \lor \neg b \lor \neg c), (\neg b \lor \neg c)$

Proposition

Unit resolution is refutation-complete for Horn clauses.

Proof idea.

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Consider $R_{II}^*(\Delta)$ of Horn clause set Δ . We have to show that if $\square \notin R_{II}^*(\Delta)$, then $\Delta (\equiv R_{II}^*(\Delta))$ is satisfiable.

- Assign true to all unit clauses in $R_{II}^*(\Delta)$.
- Those clauses that do not contain a literal I such that $\{I\}$ is one of the unit clauses have at least one negative literal.

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- Assign true to these literals.
- Results in satisfying truth assignment for $R_U^*(\Delta)$ (and $\Delta \subseteq R_{II}^*(\Delta)$).

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