Principles of AI Planning
15. Strong nondeterministic planning

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January 16th, 2013
In this chapter, we will consider the simplest case of nondeterministic planning by restricting attention to strong plans.
Concepts
Recall the definition of strong plans:

**Definition (strong plan)**

Let $S$ be the set of states of a planning task $\Pi$. Then a strong plan for $\Pi$ is a function $\pi : S_\pi \rightarrow O$ for some subset $S_\pi \subseteq S$ such that:

- $\pi(s)$ is applicable in $s$ for all $s \in S_\pi$,
- $S_\pi(s_0) \subseteq S_\pi \cup S_*$ ($\pi$ is closed),
- $S_\pi(s') \cap S_* \neq \emptyset$ for all $s' \in S_\pi(s_0)$ ($\pi$ is proper), and
- there is no state $s' \in S_\pi(s_0)$ such that $s'$ is reachable from $s'$ following $\pi$ in a strictly positive number of steps ($\pi$ is acyclic).
**Execution of a strong plan**

1. Determine the current state $s$.
2. If $s$ is a goal state then terminate.
3. Execute action $\pi(s)$.
4. Repeat from first step.
Strong plans

- Concepts
- Strong plans
- Images
- Weak preimages
- Strong preimages

Algorithms

Summary

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Strong plans

- (pick-up A B)

- (put-on A C)

- (pick-up-from-table A)
Strong plans

- Concepts
  - Strong plans
  - Images
  - Weak preimages
  - Strong preimages

- Algorithms

Summary

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Strong plans

(pick-up A B)

(pick-up-from-table A)

(put-on A C)
The **image** of a set $T$ of states with respect to an operator $o$ is the set of those states that can be reached by executing $o$ in a state in $T$. 
Images

Definition (image of a state)

\[ \text{img}_o(s) = \{s' \in S | s \xrightarrow{o} s'\} = \text{app}_o(s) \]

Definition (image of a set of states)

\[ \text{img}_o(T) = \bigcup_{s \in T} \text{img}_o(s) \]
Weak preimages

The weak preimage of a set $T$ of states with respect to an operator $o$ is the set of those states from which a state in $T$ can be reached by executing $o$.

$\text{wpreimg}_o(T)$

$T$
Weak preimages

**Definition (weak preimage of a state)**

$$wpreimg_o(s') = \{ s \in S | s \xrightarrow{o} s' \}$$

**Definition (weak preimage of a set of states)**

$$wpreimg_o(T) = \bigcup_{s \in T} wpreimg_o(s).$$
The strong preimage of a set $T$ of states with respect to an operator $o$ is the set of those states from which a state in $T$ is always reached when executing $o$.
Definition (strong preimage of a set of states)

\[ \text{spreimg}_o(T) = \{ s \in S \mid \exists s' \in T : s \xrightarrow{o} s' \land \text{img}_o(s) \subseteq T \} \]
Algorithms
Algorithms for strong planning

1 Dynamic programming (backward)
Compute operator/distance/value for a state based on the operators/distances/values of its all successor states.

1 Zero actions needed for goal states.
2 If states with $i$ actions to goals are known, states with $\leq i + 1$ actions to goals can be easily identified.

Automatic reuse of plan suffixes already found.

2 Heuristic search (forward)
Strong planning can be viewed as AND/OR graph search.

OR nodes: Choice between operators
AND nodes: Choice between effects

Heuristic AND/OR search algorithms: AO*, Proof Number Search, ...
Planning by dynamic programming

If for all successors of state $s$ with respect to operator $o$ a plan exists, assign operator $o$ to $s$.

- **Base case $i = 0$:** In goal states there is nothing to do.
- **Inductive case $i \geq 1$:** If $\pi(s)$ is still undefined and there is $o \in O$ such that for all $s' \in \text{img}_o(s)$, the state $s'$ is a goal state or $\pi(s')$ was assigned in an earlier iteration, then assign $\pi(s) = o$.

Backward distances

If $s$ is assigned a value on iteration $i \geq 1$, then the **backward distance** of $s$ is $i$. The dynamic programming algorithm essentially computes the **backward distances** of states.
Backward distances

Example

distance to $G$

$\infty$

3 2 1 0

$G$

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Definition (backward distance sets)

Let \( G \) be a set of states and \( O \) a set of operators. The \textbf{backward distance sets} \( D_i^{bwd} \) for \( G \) and \( O \) consist of those states for which there is a guarantee of reaching a state in \( G \) with at most \( i \) operator applications using operators in \( O \):

\[
D_0^{bwd} := G
\]

\[
D_i^{bwd} := D_{i-1}^{bwd} \cup \bigcup_{o \in O} spreimg_o(D_{i-1}^{bwd}) \text{ for all } i \geq 1
\]
Definition (backward distance)

Let $G$ be a set of states and $O$ a set of operators, and let $D_{bwd}^0, D_{bwd}^1, \ldots$ be the backward distance sets for $G$ and $O$. Then the **backward distance** of a state $s$ for $G$ and $O$ is

$$\delta_{bwd}^G(s) = \min\{i \in \mathbb{N} \mid s \in D_{bwd}^i\}$$

(where $\min \emptyset = \infty$).
Strong plans based on distances

Let $\Pi = \langle V, I, O, \gamma \rangle$ be a nondeterministic planning task with state set $S$ and goal states $S_*$. 

**Extraction of a strong plan from distance sets**

1. Let $S' \subseteq S$ be those states having a finite backward distance for $G = S_*$ and $O$.
2. Let $s \in S'$ be a state with distance $i = \delta_{G}^{bwd}(s) \geq 1$.
3. Assign to $\pi(s)$ any operator $o \in O$ such that $\text{img}_o(s) \subseteq D_{i-1}^{bwd}$. Hence $o$ decreases the backward distance by at least one.

Then $\pi$ is a strong plan for $\mathcal{T}$ iff $I \in S'$.

**Question:** What is the worst-case runtime of the algorithm?

**Question:** What is the best-case runtime of the algorithm if most states have a finite backward distance?
Making the algorithm a logic-based algorithm

- An algorithm that represents the states *explicitly* stops being feasible at about $10^8$ or $10^9$ states.
- For planning with bigger transition systems structural properties of the transition system have to be taken advantage of.
- As before, representing state sets as propositional formulae (or BDDs) often allows taking advantage of the structural properties: a formula (or BDD) that represents a set of states or a transition relation that has certain regularities may be very small in comparison to the set or relation.
- In the following, we will present an algorithm using a boolean-formula representation (without going into the details of how to implement it using BDDs).
Remark: The following algorithm assumes a propositional representation of the state space as opposed to a finite-domain representation. We have already seen how to translate an FDR encoding into a propositional encoding in Chapter 9 (cf. definition of the “induced propositional planning task”). Therefore, for the rest of the present section, we will assume without loss of generality that all $v \in V$ are propositional variables with domain $\mathcal{D}_v = \{0, 1\}$. 
Breadth-first search with progression and state sets (deterministic case)

Progression breadth-first search

```python
def bfs-progression(V, I, O, γ):
    goal := formula-to-set(γ)
    reached := {I}
    loop:
        if reached ∩ goal ≠ ∅:
            return solution found
        new-reached := reached ∪ ⋃_{o∈O} img_o(reached)
        if new-reached = reached:
            return no solution exists
        reached := new-reached
```

/~> This can easily be transformed into a regression algorithm.
Breadth-first search with regression and state sets (deterministic case)

Regression breadth-first search

```python
def bfs-regression(V, I, O, γ):
    init := I
    reached := formula-to-set(γ)
    loop:
        if init ∈ reached:
            return solution found
        new-reached := reached ∪ ∪ o∈O wpreimg_o(reached)
        if new-reached = reached:
            return no solution exists
        reached := new-reached
```

This algorithm is very similar to the dynamic programming algorithm for the nondeterministic case!
Breadth-first search with regression and state sets (strong nondeterministic case)

Regression breadth-first search

```python
def bfs-regression(V, I, O, γ):
    init := I
    reached := formula-to-set(γ)
    loop:
        if init ∈ reached:
            return solution found
        new-reaching := reached ∪ ∪o∈O spreimgo(reached)
        if new-reaching = reached:
            return no solution exists
        reached := new-reaching
```

- How do we define `spreimg` with logic (or BDD) operations?
Transition formula for nondeterministic operators

Let $V$ be the set of state variables and $V' := \{v' \mid v \in V\}$ a set of primed copies of the variables in $V$. Intuition:

- Variables in $V$ describe the current state $s$.
- Variables in $V'$ describe the next state $s'$.

We would like to define a formula $\tau_V(o)$ that describes the transitions labeled with $o$ between states $s$ (over $V$) and $s'$ (over $V'$) in terms of $V$ and $V'$. 
Transition formula for nondeterministic operators

The formula $\tau_V(o)$ must express

- the conditions for applicability of $o$,
- how $o$ changes state variables, and
- which state variables $o$ does not change.

A significant difficulty lies in the third requirement because different variables may be affected depending on nondeterministic choices.
Transition formula for nondeterministic operators

\[ \tau_V(o) \text{ for deterministic operators } o = \langle \chi, e \rangle \]

\[
\begin{align*}
\tau_V(o) &= \chi \land \bigwedge_{v \in V} ((EPC_v(e) \lor (v \land \neg EPC_{\neg v}(e))) \leftrightarrow v') \\
&\quad \land \bigwedge_{v \in V} (\neg (EPC_v(e) \land EPC_{\neg v}(e)))
\end{align*}
\]

Assume that \( e = \bigwedge_{a \in A} a \land \bigwedge_{d \in D} \neg d \) for \( A = \{a_1, \ldots, a_k\} \) and \( D = \{d_1, \ldots, d_l\} \) with \( A \cap D = \emptyset \). Then this becomes simpler.

\[ \tau_V(o) \text{ for STRIPS operators } o = \langle \chi, \bigwedge_{a \in A} a \land \bigwedge_{d \in D} \neg d \rangle \]

\[
\begin{align*}
\tau_V(o) &= \chi \land \bigwedge_{a \in A} a' \land \bigwedge_{d \in D} \neg d' \land \bigwedge_{v \in V \setminus (A \cup D)} (v \leftrightarrow v')
\end{align*}
\]
Transition formula for nondeterministic operators

For nondeterministic operators \( o = \langle \chi, \{ e_1, \ldots, e_n \} \rangle \) with corresponding add and delete lists \( A_i \) and \( D_i \) of \( e_i \) such that \( A_i \cap D_i = \emptyset, \ i = 1, \ldots, n \), we get:

\[
\tau_V(o) = \chi \land \bigvee_{i=1}^{n} \left( \bigwedge_{a \in A_i} a' \land \bigwedge_{d \in D_i} \neg d' \land \bigwedge_{v \in V \setminus (A_i \cup D_i)} (v \leftrightarrow v') \right)
\]

Example

Let \( V = \{ a, b \} \), \( V' = \{ a', b' \} \), and \( o = \langle \neg a, \{ a, a \land \neg b \} \rangle \). Then

\[
\tau_V(o) = \neg a \land \left( (a' \land (b \leftrightarrow b')) \lor (a' \land \neg b') \right).
\]
Computing strong preimages

Definition (substitution)

Let $\varphi, t_1, \ldots, t_n$ be propositional formulas and $\nu_1, \ldots, \nu_n$ atomic propositions.

We denote the formula obtained from $\varphi$ by simultaneous replacement of all variables $\nu_i$ by the corresponding formulas $t_i$, $i = 1, \ldots, n$, by $\varphi[t_1, \ldots, t_n/\nu_1, \ldots, \nu_n]$. 
Computing strong preimages

Definition (existential abstraction)

Let $\varphi$ be a propositional formula and $v_1, \ldots, v_n$ be atomic propositions. Then the existential abstraction of $\varphi$ wrt. $v_1, \ldots, v_n$ is recursively defined as follows:

$$\exists v. \varphi := \varphi[\top/v] \lor \varphi[\bot/v]$$

$$\exists v_1 \ldots \exists v_n. \varphi := \exists v_1 \ldots \exists v_{n-1}. (\varphi[\top/v_n] \lor \varphi[\bot/v_n])$$

For a set of variables $V = \{v_1, \ldots, v_n\}$ we use the abbreviation

$$\exists V. \varphi := \exists v_1 \ldots \exists v_n. \varphi.$$  

Note: Even with intermediate formula simplifications this can lead to an exponential blowup. BDDs can be useful here.
Computing strong preimages

Strong preimages

\[
spreimg_o(T) = \{ s \in S \mid \exists s' \in T : s \rightarrow s' \land img_o(s) \subseteq T \}
= \{ s \in S \mid (\exists s' \in S : s \rightarrow s' \land s' \in T) \land
\{ s' \in S \mid s \rightarrow s' \} \subseteq T \}
= \{ s \in S \mid (\exists s' \in S : s \rightarrow s' \land s' \in T) \land
(\forall s' \in S : s \rightarrow s' \Rightarrow s' \in T) \}
= \{ s \in S \mid (\exists s' \in S : s \rightarrow s' \land s' \in T) \land
(\neg \exists s' \in S : s \rightarrow s' \land \neg (s' \in T)) \}
\]
Computing strong preimages with boolean function operations

\[ \text{spreimg}_o(T) = \{ s \in S \mid (\exists s' \in S : s \xrightarrow{o} s' \land s' \in T) \land \\
(\neg \exists s' \in S : s \xrightarrow{o} s' \land \neg (s' \in T)) \} \]

**Strong preimages with boolean functions**

For formula \( \varphi \) characterizing set \( T \) of strongly backward-reached states:

\[ \text{spreimg}_o(\varphi) = (\exists V'. (\tau_V(o) \land \varphi[v'_1, \ldots, v'_n/v_1, \ldots, v_n])) \land \\
(\neg \exists V'. (\tau_V(o) \land \neg \varphi[v'_1, \ldots, v'_n/v_1, \ldots, v_n])) \]

We can use this regression formula for efficient **symbolic** regression search. BDDs support all necessary operations (atomic propositions, \( \neg \), \( \land \), \( \lor \), substitution, \( \exists \), \ldots).
Computing strong preimages with boolean function operations

**Example**

Let \( V = \{a, b\} \), \( V' = \{a', b'\} \), and

\[
o = \langle \neg a, \{a, a \land \neg b\} \rangle, \quad \text{i.e.,}
\]

\[
\tau_V(o) = \neg a \land \left( (a' \land (b \leftrightarrow b')) \lor (a' \land \neg b') \right).
\]

Moreover, let \( \varphi = a \). Then

\[
spreimg_o(\varphi) = \exists a' \exists b'. \left( \neg a \land \left( (a' \land (b \leftrightarrow b')) \lor (a' \land \neg b') \right) \land a' \right) \land \neg \exists a' \exists b'. \left( \neg a \land \left( (a' \land (b \leftrightarrow b')) \lor (a' \land \neg b') \right) \land \neg a' \right)
\]

\[
\equiv \neg a
\]
Progression Search

- We saw a generalization of regression search to strong planning.
- However, this search is uninformed (breadth-first search).
- Is there an analogue to A* search for strong planning?
- Yes: AO* search
  - Progression search (like A*)
  - Guided by a heuristic (like A*)
  - Guaranteed optimality (under certain conditions, like A*)
AND/OR search
AND/OR search
We describe AO* on a graph representation without intermediate nodes, i.e., as in the first figure.

There are different variants of AO*, depending on whether the graph that is being searched is an AND/OR tree, an AND/OR DAG, or a general, possibly cyclic, AND/OR graph.

The graphs we want to search, \( T(\Pi) \), are in general cyclic.

However, AO* becomes a bit more involved when dealing with cycles, so we only discuss AO* under the assumption of acyclicity and leave the generalization to cyclic state spaces as an exercise.
The search is over $\mathcal{T}(\Pi)$.

For ease of presentation, we do not distinguish between states of $\mathcal{T}(\Pi)$ and search nodes.

Also, for ease of presentation, we do not handle the case that no strong plan exists.
AO* Search

Definition (solution graph)

A solution graph for a nondeterministic transition system \( \mathcal{T} = \langle S, L, T, s_0, S_* \rangle \) is an acyclic subgraph of \( \mathcal{T} \) (viewed as a graph), \( \mathcal{T}' = \langle S', L, T' \rangle \), such that

- \( s_0 \in S' \),
- for each \( s' \in S' \setminus S_* \), there is exactly one label \( l \in L \) s.t.
  - \( T' \) contains at least one outgoing transition from \( s' \) labeled with \( l \),
  - \( T' \) contains all outgoing transitions from \( s' \) labeled with \( l \) (and \( S' \) contains the states reached via such transitions),
  - \( T' \) contains no outgoing transitions from \( s' \) labeled with any \( \tilde{l} \neq l \), and
- every directed path in \( \mathcal{T}' \) terminates at a goal state.
AO* Search

Conceptually, there are three graphs/transition systems:

- The induced transitions system $T = T(\Pi)$, which only exists as a mathematical object, but is in general not made explicit completely during AO* search,
- The current portion of $T$ explicitly represented by the search algorithm, $T_e$, and
- The current portion of $T_e$ considered by the algorithm as the cheapest/best current partial solution graph, $T_p$. 
Definition (partial solution graph)

A partial solution graph for a nondeterministic transition system \( \mathcal{T} = \langle S, L, T, s_0, S_\star \rangle \) is an acyclic subgraph of \( \mathcal{T} \) (viewed as a graph), \( \mathcal{T}_p = \langle S_p, L, T_p \rangle \), s.t.

- \( s_0 \in S_p \),
- for each \( s' \in S_p \) that is not an unexpanded leaf node in \( \mathcal{T}_p \) there is exactly one label \( l \in L \) such that
  - \( T_p \) contains at least one outgoing transition from \( s' \) labeled with \( l \),
  - \( T_p \) contains all outgoing transitions from \( s' \) labeled with \( l \) (and \( S_p \) contains the states reached via such transitions),
  - \( T_p \) contains no outgoing transitions from \( s' \) labeled with any \( \tilde{l} \neq l \), and
- every directed path in \( \mathcal{T}_p \) terminates at a goal state or an unexpanded leaf node in \( \mathcal{T}_p \).
AO* Search

Definition (cost of a partial solution graph)

Let \( h : S \rightarrow \mathbb{N} \cup \{\infty\} \) be a heuristic function for the state space \( S \) of \( T \), and let \( T_p = \langle S_p, L, T_p \rangle \) be a partial solution graph. The cost labeling of \( T_p \) is the solution to the following system of equations over the states \( S_p \) of \( T_p \):

\[
f(s) = \begin{cases} 
0 & \text{if } s \text{ is a goal state} \\
h(s) & \text{if } s \text{ is an unexpanded non-goal} \\
1 + \max_{s \rightarrow s'} f(s') & \text{for the unique outgoing action } o \text{ of } s \text{ in } T_p, \text{ otherwise.}
\end{cases}
\]

The cost of \( T_p \) is the cost labeling of its root.

AO* search keeps track of a cheapest partial solution graph by marking for each expanded state \( s \) an outgoing action \( o \) minimizing \( 1 + \max_{s \rightarrow s'} f(s') \).
AO* Search

Procedure ao-star

def ao-star(\mathcal{T}):
    let \mathcal{T}_e initially consist of the initial state s_0.
    
    while \mathcal{T}_p has unexpanded non-goal node:
        expand unexpanded non-goal node s of \mathcal{T}_p
        add new successor states to \mathcal{T}_e
        for all new states s' added to \mathcal{T}_e:
            f(s') \leftarrow h(s')
        Z \leftarrow s and its ancestors in \mathcal{T}_e along marked actions.
        while Z is not empty:
            remove from Z a state s w/o descendant in Z.
            f(s) \leftarrow \min_o \text{ applicable in } s (1 + \max_{s \rightarrow s'} f(s')).
            mark the best outgoing action for s
            (this may implicitly change \mathcal{T}_p).
    return an optimal solution graph.
Correctness (proof sketch)

- Solution graphs directly correspond to strong plans.
- Algorithm eventually terminates (finite number of possible node expansions).
- Acyclicity guarantees that extraction of $T_p$ and dynamic programming back-propagation of $f$ values always terminates.
- Marking makes sure that existing solutions are eventually marked.
AO* Search

Details

- Pseudocode omits **bookkeeping of solved states** (can improve performance).
- Choice of unexpanded non-goal node of best partial solution graph is unspecified.
  - Correctness/optimality not affected.
  - One possibility: choose node with lowest cost estimate.
  - Alternative: expand several nodes simultaneously.

- Algorithm can be extended to deal with **cycles in the AND/OR graph**.
AO* Search

Example

AO* Search

Concepts

Algorithms

Regression

Efficient implementation of regression

Progression

Summary
AO* Search

Example
AO* Search

Example

10 → 3 → 2 → 3 → 4 → 8

10 → 3 → 2 → 3 → 8
AO* Search

Example

AO* Search

Example

AO* Search

Example
AO* Search

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AO* Search

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AO* Search

Example
Heuristic Evaluation Function

- **Desireable:** informative, domain-independent heuristic to initialize cost estimates.
- Heuristic should estimate (strong) goal distances.
- Heuristic does not necessarily have to be admissible (unless we seek optimal solutions).
- We can adapt many heuristics we already know from classical planning (details omitted).
Summary
Summary

- We have considered the special case of nondeterministic planning where
  - planning tasks are fully observable and
  - we are interested in strong plans.
- We have introduced important concepts also relevant to other variants of nondeterministic planning such as
  - images and
  - weak and strong preimages.
- We have discussed some basic classes of algorithms:
  - backward induction by dynamic programming, and
  - forward search in AND/OR graphs.