Exercise 5.1 \((h^+ \text{ heuristic, } 2+2 \text{ points})\)

A 15-puzzle planning task \(\Pi = (A, I, O, \gamma)\) is given as

\[
A = \{empty(p_{i,j}) \mid 0 \leq i, j \leq 3\} \cup \{at(t_k, p_{i,j}) \mid 0 \leq i, j \leq 3, 0 \leq k \leq 14\},
\]
\[
O = \{move(t_m, p_{i,j}, p_{k,l}) \mid 0 \leq i, j, k, l \leq 3, 0 \leq m \leq 14, (i = k \text{ and } |j - l| = 1) \text{ or } (j = l \text{ and } |i - k| = 1)\},
\]
\[
\gamma = \bigwedge_{0 \leq m \leq 14} at(t_m, p_{m/4,m\%4}).
\]

Action \(move(t_m, p_{i,j}, p_{k,l})\) moves tile \(t_m\) from position \(p_{i,j}\) to position \(p_{k,l}\):

\[
move(t_m, p_{i,j}, p_{k,l}) = (at(t_m, p_{i,j}) \land empty(p_{k,l}),
\]
\[
\quad at(t_m, p_{k,l}) \land empty(p_{i,j}) \land \neg at(t_m, p_{i,j}) \land \neg empty(p_{k,l}))
\]

A syntactically possible state is \textit{legal} if each tile \(t_m\) is at some position \(p_{i,j}\), if no two tiles are at the same position and if the remaining position is the only one that is \textit{empty}. The initial state is an arbitrary state that is legal.

One possible heuristic for the 15-puzzle is the Manhattan-distance heuristic \(h^{\text{Manhattan}}\). It sums the Manhattan distances of all tiles from their current positions to their target positions, where the Manhattan distance between position \(p_{i,j}\) and \(p_{k,l}\) is given as \(|i - k| + |j - l|\).

The \(h^+\) heuristic estimates the distance of state \(s\) to the closest goal state as the length of the optimal plan in the relaxed planning task (with initial state \(s\)).

(a) Show that \(h^+(s) \geq h^{\text{Manhattan}}(s)\) for each legal state \(s\) of a 15-puzzle planning task.

(b) Show that \(h^+(s) > h^{\text{Manhattan}}(s)\) for at least one state \(s\) of a 15-puzzle planning task.

Exercise 5.2 (Relaxed planning graph and heuristics, 1+3+1+1 points)

Consider the relaxed planning task \(\Pi^+\) with variables \(A = \{a, b, c, d, e\}\), operators \(O = \{o_1, o_2\}\), \(o_1 = \{a \lor b, c \land d \land (c \lor b)\}\), \(o_2 = \{d, e\}\), goal \(\gamma = b \land e\) and initial state \(s = \{a \mapsto 1, b \mapsto 0, c \mapsto 0, d \mapsto 0, e \mapsto 0\}\).

(a) Calculate \(h^+(s)\) for \(\Pi^+\) and explain your answer.

(b) Draw the relaxed planning graph \(RPG_k(\Pi^+)\) for depth \(k = 2\) and calculate the truth values of the nodes.

(c) Calculate \(h_{\text{max}}\) for \(\Pi^+\) and explain your answer.

(d) Calculate \(h_{\text{add}}\) for \(\Pi^+\) and explain your answer.

\textit{Hint:} If you annotate the relaxed planning graph with numbers representing heuristic values (or truth values) to explain an answer, please indicate clearly the meaning of each number.

Note: The exercise sheets may and should be worked on in groups of two students. Please state both names on your solution (this also holds for submissions by e-mail).