Exercise 13.1 (Runtime, 2 marks)
You have implemented an algorithm that needs exactly $f(n)$ steps to terminate, where $n$ is the size of the input. Assume that on your machine each step takes $1\mu s$.
For which maximal input size does your algorithm terminate within one day? Which input size can it maximally process in 10 days? Answer these (two!) questions for the following runtimes:

(a) $f(n) = n$
(b) $f(n) = n^2$
(c) $f(n) = 2^n$
(d) $f(n) = n^2 + n$
(e) $f(n) = n \log n$

(this question is optional, so you do not need to answer it to receive full marks.)

Exercise 13.2 (Big-O, 2 + 1 marks)
Consider the Turing machine below. The input alphabet is $\Sigma = \mathbb{N} = \{1, 2, 3, \ldots\}$. The operator $|w|$ denotes the length of the string $w$, the relation $<$ is the smaller relation on the natural numbers.

\[
M = \text{"On input string $w$":}
\text{for } i = 1 \text{ to } |w| \\
\text{for } j = |w| \text{ downto } i + 1 \\
\text{if } w_j < w_{j-1} \text{ swap } w_j \text{ and } w_{j-1} \\
\text{endif} \\
\text{endfor} \\
\text{endfor}
\]
Assume that the runtime of a swap and of a comparison of two natural numbers is constant.

(a) What is the smallest integer $k$ such that the runtime of the Turing machine $M$ is in $O(|w|^k)$? Justify your answer.

(b) What does $M$ compute (i.e. what is written on the tape when $M$ halts)?

Exercise 13.3 (Big-O, 1 + 2 + 1 +1 marks)
Characterize the relationship between $f(n)$ and $g(n)$ in the following examples using the $O, \Theta$ or $\Omega$-notation.

1) $f(n) = n^{0.99998}$ \hspace{1cm} $g(n) = \sqrt{n}$
2) $f(n) = 2^{\log^2(n)}$ \hspace{1cm} $g(n) = \sum_{k=1}^{n^2} \frac{n}{2^k}$
3) $f(n) = n \cdot \log_2 n$ \hspace{1cm} $g(n) = \sqrt[n]{n}$
4) $f(n) = \sqrt{n}$ \hspace{1cm} $g(n) = 1000n$