Strategic Game

• A strategic game $G$ consists of
  – a finite set $N$ (the set of players)
  – for each player $i \in N$ a non-empty set $A_i$ (the set of actions or strategies available to player $i$), whereby $A = \prod_i A_i$
  – for each player $i \in N$ a function $u_i : A \to R$ (the utility or payoff function)
  – $G = (N, (A_i), (u_i))$

• If $A$ is finite, then we say that the game is finite
Playing the Game

• Each player $i$ makes a decision which action to play: $a_i$
• All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, ..., a_n)$
• Then each player gets the payoff $u_i(a^*)$
• Of course, each player tries to maximize its own payoff, but what is the right decision?
• **Note**: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  – If we want to model something like this, the payoff function must be changed
**Notation**

- For *2-player games*, we use a matrix, where the strategies of **player 1** are the *rows* and the strategies of **player 2** the *columns*.

- The payoff for every action profile is specified as a pair \( x, y \), whereby \( x \) is the value for player 1 and \( y \) is the value for player 2.

- Example: For \((T,R)\), **player 1** gets \( x_{12} \), and **player 2** gets \( y_{12} \).

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th>Player 2</th>
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<tbody>
<tr>
<td><strong>L</strong> action</td>
<td>( x_{11}, y_{11} )</td>
<td>( x_{12}, y_{12} )</td>
</tr>
<tr>
<td><strong>R</strong> action</td>
<td>( x_{21}, y_{21} )</td>
<td>( x_{22}, y_{22} )</td>
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</tbody>
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18/4
Example Game: Bach and Stravinsky

- Two people want to go out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the Battle of the Sexes

<table>
<thead>
<tr>
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<th>Stravinsky</th>
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<tr>
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<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>
Example Game: Hawk-Dove

• Two animals fighting over some prey.
• Each can behave like a dove or a hawk
• The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
• This game is also called chicken.

<table>
<thead>
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<tr>
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<td>3,3</td>
<td>1,4</td>
</tr>
<tr>
<td>Hawk</td>
<td>4,1</td>
<td>0,0</td>
</tr>
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Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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<td>0,4</td>
</tr>
<tr>
<td>Confess</td>
<td>4,0</td>
<td>1,1</td>
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</tbody>
</table>
The 2/3 of Average Game

• You have $n$ players that are allowed to choose a number between 1 and 100.
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would you play?
Solving a Game

• What is the right move?
• Different possible solution concepts
  – Elimination of strictly or weakly dominated strategies
  – Maximin strategies (for minimizing the loss in zero-sum games)
  – Nash equilibrium
• How difficult is it to compute a solution?
• Are there always solutions?
• Are the solutions unique?
Strictly Dominated Strategies

• Notation:
  - Let $a = (a_i)$ be a strategy profile
  - $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
  - $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots a_n)$

• Strictly dominated strategy:
  - An strategy $a_j^* \in A_j$ is strictly dominated if there exists a strategy $a'_j$ such that for all strategy profiles $a \in A$:
    \[
    u_j(a_{-j}, a'_j) > u_j(a_{-j}, a_j^*)
    \]

• Of course, it is not rational to play strictly dominated strategies
Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can **eliminate** them from the game.
- This can be done **iteratively**.
- If this converges to a single strategy profile, the result is **unique**.
- This can be regarded as the **result** of the game, because it is the **only** rational outcome.
Iterated Elimination: Example

• Eliminate:
  , dominated by
  , dominated by
  , dominated by
  , dominated by
  , dominated by

➢ Result:

<table>
<thead>
<tr>
<th></th>
<th>b1</th>
<th>b2</th>
<th>b3</th>
<th>b4</th>
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</thead>
<tbody>
<tr>
<td>a1</td>
<td>1,7</td>
<td>2,5</td>
<td>7,2</td>
<td>0,1</td>
</tr>
<tr>
<td>a2</td>
<td>5,2</td>
<td>3,3</td>
<td>5,2</td>
<td>0,1</td>
</tr>
<tr>
<td>a3</td>
<td>7,0</td>
<td>2,5</td>
<td>0,4</td>
<td>0,1</td>
</tr>
<tr>
<td>a4</td>
<td>0,0</td>
<td>0,-2</td>
<td>0,0</td>
<td>9,-1</td>
</tr>
</tbody>
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18/12
Iterated Elimination: Prisoner’s Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option.
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well.
- So, they both confess.

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Weakly Dominated Strategies

• Instead of strict domination, we can also go for weak domination:
  – An strategy \( a_j^* \in A_j \) is \textit{weakly dominated} if there exists a strategy \( a_j' \) such that for all strategy profiles \( a \in A \):
    \[
    u_j(a_{-j}, a_j') \geq u_j(a_{-j}, a_j^*)
    \]
    and for at least one profile \( a \in A \):
    \[
    u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*).
    \]
Results of Iterative Elimination of Weakly Dominated Strategies

• The result is not necessarily unique

• Example:
  - Eliminate
    - Eliminate:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
<td>1,1</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,1</td>
</tr>
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</table>
Analysis of the Guessing 2/3 of the Average Game

• All strategies above 67 are weakly dominated, since if you win with >67, you will also be able to win with 67, so they can be eliminated!
• This means, that all strategies above $\frac{2}{3} \times 67$
  can be eliminated
• ... and so on
• ... until all strategies above 1 have been eliminated!
• So: The rationale strategy would be to play 1!
If there is no Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?

Nash equilibrium

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Nash Equilibrium

- A *Nash equilibrium* is an action profile $a^* \in A$ with the property that for all players $i \in N$:
  $$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \ \forall \ a_i \in A_i$$

- In words, it is an action profile such that there is no incentive for any agent to deviate from it.

- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept.

- If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a *Nash equilibrium*.
Example Nash-Equilibrium: Prisoner’s Dilemma

- Don’t – Don’t
  - not a NE
- Don’t – Confess
  (and vice versa)
  - not a NE
- Confess – Confess
  - NE

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Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove: not a NE
- Hawk-Hawk: not a NE
- Dove-Hawk: is a NE
- Hawk-Dove: is, of course, another NE
- So, NEs are not necessarily unique

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Auctions

• An object is to be assigned to a player in the set \( \{1, \ldots, n\} \) in exchange for a payment.
• Players' valuation of the object is \( v_i \), and \( v_1 > v_2 > \ldots > v_n \).
• The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers).
• The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment.
• The payment for a first price auction is the highest bid.
• What are the Nash equilibria in this case?
Formalization

- Game $G = (\{1, \ldots, n\}, (A_i), (u_i))$
- $A_i$: bids $b_i \in \mathbb{R}^+$
- $u_i(b_{-i}, b_i) = v_i - b_i$ if $i$ has won the auction, 0 otherwise
- Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.
Nash Equilibria for First-Price Sealed-Bid Auctions

• The Nash equilibria of this game are all profiles $b$ with:
  - $b_i \leq b_1$ for all $i \in \{2, ..., n\}$
    • No $i$ would bid more than $v_2$ because it could lead to negative utility
    • If a $b_i$ (with $< v_2$) is higher than $b_1$ player 1 could increase its utility by bidding $v_2 + \varepsilon$
    • So 1 wins in all NEs
  - $v_1 \geq b_1 \geq v_2$
    • Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - $b_j = b_1$ for at least one $j \in \{2, ..., n\}$
    • Otherwise player 1 could have gotten the object for a lower bid
Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game.
- No NE at all! What shall we do here?

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</thead>
<tbody>
<tr>
<td>Head</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
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<td>-1,1</td>
<td>1,-1</td>
</tr>
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Randomizing Actions ...

- Since there does not seem to exist a rational decision, it might be best to randomize strategies.
- Play **Head** with probability $p$ and **Tail** with probability $1-p$
- Switch to expected utilities

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</tr>
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Some Notation

• Let $G = (N, (A_i), (u_i))$ be a strategic game
• Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$ – the set of mixed strategies $\alpha_i \in \Delta(A_i)$
• $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$
• A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \prod_i \alpha_i(a_i)$
• The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) u_i(a)$
Example of a Mixed Strategy

- Let
  - \( \alpha_1(H) = 2/3, \alpha_1(T) = 1/3 \)
  - \( \alpha_2(H) = 1/3, \alpha_2(T) = 2/3 \)
- Then
  - \( p(H,H) = 2/9 \)
  - \( p(H,T) = \)
  - \( p(T,H) = \)
  - \( p(T,T) = \)
  - \( U_1(\alpha_1, \alpha_2) = \)

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Mixed Extensions

• The **mixed extension** of the strategic game \((N, (A_i), (u_i))\) is the strategic game \((N, \Delta(A_i), (U_i))\).

• The **mixed strategy Nash equilibrium of a strategic game** is a Nash equilibrium of its mixed extension.

• Note that the **Nash equilibria in pure strategies** (as studied in the last part) are just a special case of mixed strategy equilibria.
Nash’s Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is finite
- So, there *exists* always a solution
- What is the *computational complexity*?
- Identifying a NE with a value larger than a particular value is *NP-hard*
The Support

- We call all pure actions $a_i$ that are chosen with non-zero probability by $\alpha_i$ the support of the mixed strategy $\alpha_i$.

**Lemma.** Given a finite strategic game, $\alpha^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_{-i}^*$. 
Using the Support Lemma

- The **Support Lemma** can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
  - There are 4 potential Nash equilibria in pure strategies
    - *Easy to check*
  - There are another 4 potential Nash equilibrium types with a 1-support (pure) against 2-support mixed strategies
    - Exists only if one of the corresponding pure strategy profiles is already a Nash equilibrium (follows from Support Lemma)
  - There exists one other potential Nash equilibrium type with a 2-support against a 2-support mixed strategies
    - Here we can use the Support Lemma to compute an NE (if there exists one)
A Mixed Nash Equilibrium for Matching Pennies

• There is clearly no NE in pure strategies.

• Lets try whether there is a NE in mixed strategies.

• Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha_{-1}^*$.

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</table>

• $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$

• $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$

• $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$

• $\alpha_2(H) - \alpha_2(T) = -\alpha_2(H) + \alpha_2(T)$

• $2\alpha_2(H) = 2\alpha_2(T)$

• $\alpha_2(H) = \alpha_2(T)$

• Because of $\alpha_2(H) + \alpha_2(T) = 1$:
  - $\alpha_2(H) = \alpha_2(T) = 1/2$
  - Similarly for player 1!

• $U_1(\alpha^*) = 0$
Mixed NE for BoS

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<td>1,2</td>
</tr>
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</table>

- There are obviously 2 NEs in pure strategies.
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

- \( U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S))) \)
- \( U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B) + 0\alpha_2(S) \)
- \( U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B) + 1\alpha_2(S) \)
- \( 2\alpha_2(B) = 1\alpha_2(S) \)
- Because of \( \alpha_2(B) + \alpha_2(S) = 1 \):
  - \( \alpha_2(B) = 1/3 \)
  - \( \alpha_2(S) = 2/3 \)

- Similarly for player 1!
- \( U_1(\alpha^*) = 2/3 \)
The 2/3 of Average Game

• You have $n$ players that are allowed to choose a number between 1 and $K$.
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would you play?
• What mixed strategy would you play?
A Nash Equilibrium in Pure Strategies

- All playing 1 is a NE in pure strategies
  - A deviation does not make sense
- All playing the same number different from 1 is **not a NE**
  - Choosing the number just below gives you more
- Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.
- So: *Why did you not choose 1?*
- Perhaps you acted rationally by assuming that the others do not act rationally?
Are there Proper Mixed Strategy Nash Equilibria?

• Assume there exists a mixed NE \( \alpha \) different from the pure NE \((1,1,...,1)\)
• Then there exists a maximal \( k^* > 1 \) which is played by some player with a probability \( > 0 \).
  – Assume player \( i \) does so, i.e., \( k^* \) is in the support of \( a_i \).
• This implies \( U_i(k^*,a_{-i}) > 0 \), since \( k^* \) should be as good as all the other strategies of the support.
• Let \( a \) be a realization of \( \alpha \) s.t. \( u_i(a) > 0 \). Then at least one other player must play \( k^* \), because not all others could play below \( 2/3 \) of the average!
• In this situation player \( i \) could get more by playing \( k^*-1 \).
• This means, playing \( k^*-1 \) is better than playing \( k^* \), i.e., \( k^* \) cannot be in the support, i.e., \( \alpha \) cannot be a NE
Summary

- **Strategic games** are one-shot games, where everybody plays its move simultaneously.
- Each player gets a payoff based on its payoff function and the resulting action profile.
- Iterated elimination of strictly dominated strategies is a convincing solution concept.
- **Nash equilibrium** is another solution concept: Action profiles, where no player has an incentive to deviate.
- It also might not be unique and there can be even infinitely many NEs or none at all!
- For every finite strategic game, there exists a Nash equilibrium in mixed strategies.
- Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff \( \rightsquigarrow \) Support Lemma.
- Computing a NE in mixed strategies is NP-hard.