Introduction to Multi-Agent Programming

10. Game Theory

Strategic Reasoning and Acting
Alexander Kleiner and Bernhard Nebel
Strategic Game

• A strategic game $G$ consists of
  - a finite set $N$ (the set of players)
  - for each player $i \in N$ a non-empty set $A_i$ (the set of actions or strategies available to player $i$), whereby $A = \prod_i A_i$
  - for each player $i \in N$ a function $u_i: A \rightarrow R$ (the utility or payoff function)
  - $G = (N, (A_i), (u_i))$

• If $A$ is finite, then we say that the game is finite
Playing the Game

• Each player $i$ makes a decision which action to play: $a_i$
• All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, ..., a_n)$
• Then each player gets the payoff $u_i(a^*)$
• Of course, each player tries to maximize its own payoff, but what is the right decision?
• Note: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  – If we want to model something like this, the payoff function must be changed
Notation

- For 2-player games, we use a matrix, where the strategies of **player 1** are the rows and the strategies of **player 2** the columns.
- The payoff for every action profile is specified as a pair $x,y$, whereby $x$ is the value for player 1 and $y$ is the value for player 2.
- Example: For (T,R), **player 1** gets $x_{12}$, and **player 2** gets $y_{12}$.

| Player1 | Player2
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<td><strong>T</strong></td>
<td><strong>L</strong></td>
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<tr>
<td>action</td>
<td>action</td>
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<tr>
<td>$x_{11},y_{11}$</td>
<td>$x_{12},y_{12}$</td>
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<td><strong>B</strong></td>
<td><strong>R</strong></td>
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<td>action</td>
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<tr>
<td>$x_{21},y_{21}$</td>
<td>$x_{22},y_{22}$</td>
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Example Game: Bach and Stravinsky

- Two people want to go out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

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<td>1,2</td>
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Example Game: Hawk-Dove

• Two animals fighting over some prey.
• Each can behave like a dove or a hawk
• The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
• This game is also called *chicken.*

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<td>0,0</td>
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Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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The 2/3 of Average Game

• You have $n$ players that are allowed to choose a number between 1 and 100.
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would you play?
Solving a Game

• What is the right move?
• Different possible solution concepts
  – Elimination of strictly or weakly dominated strategies
  – Maximin strategies (for minimizing the loss in zero-sum games)
  – Nash equilibrium
• How difficult is it to compute a solution?
• Are there always solutions?
• Are the solutions unique?
Strictly Dominated Strategies

• **Notation:**
  - Let $a = (a_i)$ be a strategy profile
  - $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ... a_n)$
  - $(a_{-i}, a'_i) := (a_1, ..., a_{i-1}, a'_i, a_{i+1}, ... a_n)$

• **Strictly dominated strategy:**
  - An strategy $a_j^* \in A_j$ is *strictly dominated* if there exists a strategy $a'_j$ such that for all strategy profiles $a \in A$:
    \[ u_j(a_j, a'_j) > u_j(a_j, a_j^*) \]

• Of course, it is **not rational** to play strictly dominated strategies
Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game
- This can be done iteratively
- If this converges to a single strategy profile, the result is unique
- This can be regarded as the result of the game, because it is the only rational outcome
**Iterated Elimination: Example**

- Eliminate:
  - , dominated by
  - , dominated by
  - , dominated by
  - , dominated by

- Result:

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<tr>
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<th>b1</th>
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<tbody>
<tr>
<td>a1</td>
<td>1,7</td>
<td>2,5</td>
<td>7,2</td>
<td>0,1</td>
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<tr>
<td>a2</td>
<td>5,2</td>
<td>3,3</td>
<td>5,2</td>
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<td>a3</td>
<td>7,0</td>
<td>2,5</td>
<td>0,4</td>
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<td>a4</td>
<td>0,0</td>
<td>0,-2</td>
<td>0,0</td>
<td>9,-1</td>
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Iterated Elimination: Prisoner’s Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option.
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well.
- So, they both confess.

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Weakly Dominated Strategies

• Instead of strict domination, we can also go for weak domination:
  
  – An strategy \( a_j^* \in A_j \) is weakly dominated if there exists a strategy \( a'_j \) such that for all strategy profiles \( a \in A \):
  
    \[ u_j(a_j, a'_j) \geq u_j(a_j, a_j^*) \]

    and for at least one profile \( a \in A \):
    
    \[ u_j(a_j, a'_j) > u_j(a_j, a_j^*) \].
Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate: $T (≤M)$
    - Result: $(1,1)$
  - Eliminate: $B (≤M)$
    - Result: $(2,1)$

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<tr>
<td>$T$</td>
<td>2,1</td>
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<tr>
<td>$M$</td>
<td>2,1</td>
<td>1,1</td>
</tr>
<tr>
<td>$B$</td>
<td>0,0</td>
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Analysis of the Guessing 2/3 of the Average Game

• All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!

• This means, that all strategies above \( \frac{2}{3} \times 67 \) can be eliminated

• ... and so on

• ... until all strategies above 1 have been eliminated!

• So: The *rationale* strategy would be to play 1!
If there is no Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?

- Nash equilibrium

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Nash Equilibrium

A *Nash equilibrium* is an action profile \( a^* \in A \) with the property that for all players \( i \in N \):
\[
u_i(a^*) = u_i(a^*, a^*) \geq u_i(a^*, a) \quad \forall \, a \in A_i
\]

In words, it is an action profile such that there is no incentive for any agent to deviate from it.

While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept.

If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a *Nash equilibrium*. 
Example Nash-Equilibrium: Prisoner’s Dilemma

- Don’t – Don’t  
  - not a NE  
- Don’t – Confess  
  (and vice versa)  
  - not a NE  
- Confess – Confess  
  - NE

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Example Nash-Equilibrium: Hawk-Dove

- **Dove-Dove:**
  - not a NE
- **Hawk-Hawk**
  - not a NE
- **Dove-Hawk**
  - is a NE
- **Hawk-Dove**
  - is, of course, another NE

So, NEs are not necessarily unique

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Auctions

• An object is to be assigned to a player in the set \{1,\ldots,n\} in exchange for a payment.
• Players’ valuation of the object is \(v_i\), and \(v_1 > v_2 > \ldots > v_n\).
• The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers).
• The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment.
• The payment for a first price auction is the highest bid.
• What are the Nash equilibria in this case?
Formalization

• Game $G = (\{1, \ldots, n\}, (A_i), (u_i))$

• $A_i$: bids $b_i \in R^+$

• $u_i(b_{-i}, b_i) = v_i - b_i$ if $i$ has won the auction, 0 otherwise

• Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.
Nash Equilibria for First-Price Sealed-Bid Auctions

• The Nash equilibria of this game are all profiles \( b \) with:
  - \( b_i \leq b_1 \) for all \( i \in \{2, \ldots, n\} \)
    • No \( i \) would bid more than \( v_2 \) because it could lead to negative utility
    • If a \( b_i \) (with \( < v_2 \)) is higher than \( b_1 \), player 1 could increase its utility by bidding \( v_2 + \varepsilon \)
    • So 1 wins in all NEs
  - \( v_1 \geq b_1 \geq v_2 \)
    • Otherwise, player 1 either loses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - \( b_j = b_1 \) for at least one \( j \in \{2, \ldots, n\} \)
    • Otherwise player 1 could have gotten the object for a lower bid
Another Game: Matching Pennies

• Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.

• This is also a zero-sum or strictly competitive game

• No NE at all! What shall we do here?

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<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1,1</td>
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Randomizing Actions ...

- Since there does not seem to exist a rational decision, it might be best to randomize strategies.
- Play Head with probability $p$ and Tail with probability $1-p$.
- Switch to expected utilities.

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Some Notation

• Let $G = (N, (A_i), (u_i))$ be a strategic game
• Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$ – the set of mixed strategies $\alpha_i \in \Delta(A_i)$
• $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$
• A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \prod_i \alpha_i(a_i)$
• The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) u_i(a)$
Example of a Mixed Strategy

- Let
  - $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$
  - $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$

- Then
  - $p(H,H) = 2/9$
  - $p(H,T) =$
  - $p(T,H) =$
  - $p(T,T) =$
  - $U_1(\alpha_1, \alpha_2) =$

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Mixed Extensions

• The mixed extension of the strategic game \((N, (A_i), (u_i))\) is the strategic game \((N, \Delta(A_i), (U_i))\).

• The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.

• Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.
Nash’s Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is **finite**
- So, there **exists** always a solution
- What is the **computational complexity**?
- Identifying a NE with a value larger than a particular value is **NP-hard**
The Support

- We call all pure actions $a_i$ that are chosen with non-zero probability by $\alpha_i$ the support of the mixed strategy $\alpha_i$.

**Lemma.** Given a finite strategic game, $\alpha^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_i^*$. 
Using the Support Lemma

- The **Support Lemma** can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
  - There are 4 potential Nash equilibria in **pure strategies**
    - **Easy to check**
  - There are another 4 potential Nash equilibrium types with a **1-support** (pure) against **2-support** mixed strategies
    - Exists only if one of the corresponding pure strategy profiles is already a Nash equilibrium (follows from **Support Lemma**)
  - There exists one other potential Nash equilibrium type with a **2-support** against a **2-support** mixed strategies
    - Here we can use the **Support Lemma** to compute an NE (if there exists one)
A Mixed Nash Equilibrium for Matching Pennies

• There is clearly no NE in pure strategies
• Let's try whether there is a NE $\alpha^*$ in mixed strategies
• Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha^*_2$

- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$
- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$
- $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$
- $\alpha_2(H) - \alpha_2(T) = - \alpha_2(H) + \alpha_2(T)$
- $2\alpha_2(H) = 2\alpha_2(T)$
- $\alpha_2(H) = \alpha_2(T)$
- Because of $\alpha_2(H) + \alpha_2(T) = 1$
  - $\alpha_2(H) = \alpha_2(T) = 1/2$
  - Similarly for player 1!

- $U_1(\alpha^*) = 0$
### Mixed NE for BoS

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- There are obviously 2 NEs in pure strategies.
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

\[
U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S)))
\]

\[
U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B) + 0\alpha_2(S)
\]

\[
U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B) + 1\alpha_2(S)
\]

- \(2\alpha_2(B) = 1\alpha_2(S)\)
- Because of \(\alpha_2(B) + \alpha_2(S) = 1:\)
  - \(\alpha_2(B) = 1/3\)
  - \(\alpha_2(S) = 2/3\)

- Similarly for player 1!

\[
U_1(\alpha^* ) = 2/3
\]
The 2/3 of Average Game

• You have $n$ players that are allowed to choose a number between 1 and $K$.
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would you play?
• What mixed strategy would you play?
A Nash Equilibrium in Pure Strategies

• All playing 1 is a NE in pure strategies
  – A deviation does not make sense
• All playing the same number different from 1 is **not a NE**
  – Choosing the number just below gives you more
• Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.

• So: **Why did you not choose 1?**
• Perhaps **you acted rationally** by assuming that the **others do not act rationally**?
Are there Proper Mixed Strategy Nash Equilibria?

- Assume there exists a mixed NE $\alpha$ different from the pure NE $(1,1,...,1)$
- Then there exists a maximal $k^* > 1$ which is played by some player with a probability $> 0$.
  - Assume player $i$ does so, i.e., $k^*$ is in the support of $\alpha_i$.
- This implies $U_i(k^*, \alpha_{-i}) > 0$, since $k^*$ should be as good as all the other strategies of the support.
- Let $a$ be a realization of $\alpha$ s.t. $u_i(a) > 0$. Then at least one other player must play $k^*$, because not all others could play below $2/3$ of the average!
- In this situation player $i$ could get more by playing $k^*-1$.
- This means, playing $k^*-1$ is better than playing $k^*$, i.e., $k^*$ cannot be in the support, i.e., $\alpha$ cannot be a NE
Summary

• **Strategic games** are one-shot games, where everybody plays its move simultaneously.
• Each player gets a payoff based on its payoff function and the resulting action profile.
• **Iterated elimination of strictly dominated strategies** is a convincing solution concept.
• **Nash equilibrium** is another solution concept: Action profiles, where no player has an incentive to deviate.
• It also might not be unique and there can be even infinitely many NEs or none at all!

➢ For every finite strategic game, there exists a Nash equilibrium in mixed strategies.
• Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff \( \sim \) **Support Lemma**.
• Computing a NE in mixed strategies is NP-hard.