Introduction to Multi-Agent Programming

10. Game Theory

Strategic Reasoning and Acting

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Strategic Game

- A strategic game G consists of
  - a finite set N (the set of players)
  - for each player $i \in N$ a non-empty set A_i (the set of actions or strategies available to player i), whereby $A = \Pi_i A_i$
  - for each player $i \in N$ a function $u_i : A \rightarrow R$ (the utility or payoff function)
  - $G = (N, (A_i), (u_i))$
- If A is finite, then we say that the game is *finite*

Playing the Game

- Each player i makes a decision which action to play: a_i
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
- Then each player gets the payoff $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- Note: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed

Notation

- For *2-player games*, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns
- The payoff for every action profile is specified as a pair *x,y*, whereby *x* is the value for player 1 and *y* is the value for player 2
- Example: For (T,R), player 1 gets x_{12}, and player 2 gets y_{12}

	Player 2 **L** action	Player 2 **R** action
Player1 **T** action	x_{11},y_{11}	x_{12},y_{12}
Player1 **B** action	x_{21},y_{21}	x_{22},y_{22}

Example Game: Bach and Stravinsky

- Two people want to out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

	Bach	Stra-vinsky
Bach	2,1	0,0
Stra-vinsky	0,0	1,2

Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
- This game is also called *chicken.*

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Example Game: Prisoner's Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

	Don't confess	Confess
Don't confess	3,3	0,4
Confess	4,0	1,1

The 2/3 of Average Game

- You have *n* players that are allowed to choose a number between 1 and *100*.
- The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners
- What number would you play?

Solving a Game

- What is the right move?
- Different possible solution concepts
  - Elimination of strictly or weakly dominated strategies
  - Maximin strategies (for minimizing the loss in zero-sum games)
  - Nash equilibrium
- How difficult is it to compute a solution?
- Are there always solutions?
- Are the solutions unique?

Strictly Dominated Strategies

- Notation:
  - Let $a = (a_i)$ be a strategy profile
  - $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
  - $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots a_n)$
- Strictly dominated strategy:
  - An strategy $a_j^* \in A_j$ is *strictly dominated* if there exists a strategy a_j' such that for all strategy profiles $a \in A$:

    $u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*)$
- Of course, it is not rational to play strictly dominated strategies

Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game
- This can be done iteratively
- If this converges to a single strategy profile, the result is unique
- This can be regarded as the result of the game, because it is the only rational outcome

Iterated Elimination: Example

- Eliminate:

  , dominated by

  , dominated by

  , dominated by

  , dominated by

  , dominated by

  , dominated by

➢ Result:

	b1	b2	b3	b4
a1	1,7	2,5	7,2	0,1
a2	5,2	3,3	5,2	0,1
a3	7,0	2,5	0,4	0,1
a4	0,0	0,-2	0,0	9,-1

Iterated Elimination: Prisoner's Dilemma

- Player 1 reasons that "not confessing" is strictly dominated and eliminates this option
- Player 2 reasons that player 1 will not consider "not confessing". So he will eliminate this option for himself as well
- So, they both confess

	Don't confess	Confes s
Don't confess	3,3	0,4
Confes s	4,0	1,1

Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy $a_j^* \in A_j$ is *weakly dominated* if there exists a strategy a_j' such that for all strategy profiles $a \in A$:

    $$u_j(a_{-j}, a_j') \geq u_j(a_{-j}, a_j^*)$$

    and for at least one profile $a \in A$:

    $$u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*).$$

Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate
  - Eliminate:

	L	R
T	2,1	0,0
M	2,1	1,1
B	0,0	1,1

Analysis of the *Guessing 2/3 of the Average* Game

- All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!
- This means, that all strategies above

  2/3 x 67

  can be eliminated
- ... and so on
- ... until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!

If there is no Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?

➢ Nash equilibrium

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Nash Equilibrium

- A *Nash equilibrium* is an action profile $a^* \in A$ with the property that for all players $i \in N$:

  $$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \; \forall \; a_i \in A_i$$

- In words, it is an action profile such that there is no incentive for any agent to deviate from it
- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept
- If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium

Example Nash-Equilibrium: Prisoner's Dilemma

- Don't – Don't
  - not a NE
- Don't – Confess (and vice versa)
  - not a NE
- Confess – Confess
  - NE

	Don't confess	Confes s
Don't confess	3,3	0,4
Confes s	4,0	1,1

Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:
  - not a NE
- Hawk-Hawk
  - not a NE
- Dove-Hawk
  - is a NE
- Hawk-Dove
  - is, of course, another NE
- So, NEs are not necessarily unique

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Auctions

- An object is to be assigned to a player in the set {1,…,n} in exchange for a payment.
- Players *i* valuation of the object is v_i, and $v_1 > v_2 > \ldots > v_n$.
- The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers)
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
- The payment for a *first price* auction is the highest bid.
- What are the Nash equilibria in this case?

Formalization

- Game G = $(\{1,\ldots,n\}, (A_i), (u_i))$
- A_i: bids $b_i \in R^+$
- $u_i(b_{-i}, b_i) = v_i - b_i$ if i has won the auction, 0 othwerwise
- Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.

Nash Equilibria for First-Price Sealed-Bid Auctions

- The Nash equilibria of this game are all profiles b with:
  - $b_i \leq b_1$ for all $i \in \{2, \ldots, n\}$
    - No i would bid more than v_2 because it could lead to negative utility
    - If a b_i (with $< v_2$) is higher than b_1 player 1 could increase its utility by bidding $v_2 + \varepsilon$
    - So 1 wins in all NEs
  - $v_1 \geq b_1 \geq v_2$
    - Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - $b_j = b_1$ for at least one $j \in \{2, \ldots, n\}$
    - Otherwise player 1 could have gotten the object for a lower bid

Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game
- No NE at all! What shall we do here?

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Randomizing Actions …

- Since there does not seem to exist a rational decision, it might be best to randomize strategies.
- Play Head with probability p and Tail with probability $1-p$
- Switch to expected utilities

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Some Notation

- Let $G = (N, (A_i), (u_i))$ be a strategic game
- Then $\Delta(A_i)$ shall be the set of probability distributions over A_i – the set of mixed strategies $\alpha_i \in \Delta(A_i)$
- $\alpha_i(a_i)$ is the probability that a_i will be chosen in the mixed strategy α_i
- A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on A: $p(a) = \Pi_i \alpha_i(a_i)$
- The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a)\ u_i(a)$

Example of a Mixed Strategy

- Let
  - $\alpha_1(H) = 2/3,\ \alpha_1(T) = 1/3$
  - $\alpha_2(H) = 1/3,\ \alpha_2(T) = 2/3$
- Then
  - $p(H,H) = 2/9$
  - $p(H,T) =$
  - $p(T,H) =$
  - $p(T,T) =$
  - $U_1(\alpha_1, \alpha_2) =$

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Mixed Extensions

- The mixed extension of the strategic game $(N, (A_i), (u_i))$ is the strategic game $(N, \Delta(A_i), (U_i))$.
- The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.
- Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.

Nash's Theorem

Theorem. Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is finite
- So, there exists always a solution
- What is the computational complexity?
- Identifying a NE with a value larger than a particular value is NP-hard

The Support

- We call all pure actions a_i that are chosen with non-zero probability by α_i the support of the mixed strategy α_i

Lemma. Given a finite strategic game**,** α^* is a *mixed strategy equilibrium* if and only if for every player *i every pure strategy in the support* of α_i^* is a best response to α_{-i}^* .

Using the Support Lemma

- The Support Lemma can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
  - There are 4 potential Nash equilibria in pure strategies
    - *Easy to check*
  - There are another 4 potential Nash equilibrium types with a 1-support (pure) against 2-support mixed strategies
    - Exists only if one of the corresponding pure strategy profiles is already a Nash equilibrium (follows from Support Lemma)
  - There exists one other potential Nash equilibrium type with a 2-support against a 2-support mixed strategies
    - Here we can use the Support Lemma to compute an NE (if there exists one)

A Mixed Nash Equilibrium for Matching Pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- There is clearly no NE in pure strategies
- Lets try whether there is a NE α^* in mixed strategies
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy α_{-1}^*

- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$

- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$
- $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$
- $\alpha_2(H) - \alpha_2(T) = -\alpha_2(H) + \alpha_2(T)$
- $2\alpha_2(H) = 2\alpha_2(T)$
- $\alpha_2(H) = \alpha_2(T)$
- Because of $\alpha_2(H) + \alpha_2(T) = 1$:

➢ $\alpha_2(H) = \alpha_2(T) = 1/2$

➢ Similarly for player 1!

❖ $U_1(\alpha^*) = 0$

Mixed NE for BoS

	Bach	Stra-vinsky
Bach	2,1	0,0
Stra-vinsky	0,0	1,2

- There are obviously 2 NEs in pure strategies
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

- $U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S)))$

- $U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B)+0\alpha_2(S)$
- $U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B)+1\alpha_2(S)$
- $2\alpha_2(B) = 1\alpha_2(S)$
- Because of $\alpha_2(B)+\alpha_2(S) = 1$:

➢ $\alpha_2(B)=1/3$

➢ $\alpha_2(S)=2/3$

➢ Similarly for player 1!

❖ $U_1(\alpha^*) = 2/3$

The 2/3 of Average Game

- You have *n* players that are allowed to choose a number between 1 and *K*.
- The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners
- What number would you play?
- What mixed strategy would you play?

A Nash Equilibrium in Pure Strategies

- All playing 1 is a NE in pure strategies
  - A deviation does not make sense
- All playing the same number different from 1 is not a NE
  - Choosing the number just below gives you more
- Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.
- So: ***Why did you not choose 1?***
- Perhaps you acted rationally by assuming that the others do not act rationally?

Are there Proper Mixed Strategy Nash Equilibria?

- Assume there exists a mixed NE α different from the pure NE (1,1,…,1)
- Then there exists a maximal $k^* > 1$ which is played by some player with a probability > 0.
  - Assume player i does so, i.e., k^* is in the support of α_i.
- This implies $U_i(k^*,\alpha_{-i}) > 0$, since k^* should be as good as all the other strategies of the support.
- Let a be a realization of α s.t. $u_i(a) > 0$. Then at least one other player must play k^*, because not all others could play below 2/3 of the average!
- In this situation player i could get more by playing k^*-1.
- This means, playing k^*-1 is better than playing k^*, i.e., k^* cannot be in the support, i.e., α cannot be a NE

Summary

- Strategic games are one-shot games, where everybody plays its move simultaneously
- Each player gets a payoff based on its payoff function and the resulting action profile.
- Iterated elimination of strictly dominated strategies is a convincing solution concept.
- Nash equilibrium is another solution concept: Action profiles, where no player has an incentive to deviate
- It also might not be unique and there can be even infinitely many NEs or none at all!
- ➢ For every finite strategic game, there exists a Nash equilibrium in mixed strategies
- Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff $\rightsquigarrow$ Support Lemma
- Computing a NE in mixed strategies is NP-hard