Exercise 13.1 (Runtime, 2 marks)
You have implemented an algorithm that needs exactly \( f(n) \) steps to terminate, where \( n \) is the size of the input. Assume that on your machine each step takes 1\( \mu \text{s} \).
For which maximal input size does your algorithm terminate within one day? Which input size can it maximally process in 10 days? Answer these (two!) questions for the following runtimes:

(a) \( f(n) = n \)
(b) \( f(n) = n \log n \)  
(this question is optional, so you do not need to answer it to receive full marks.)
(c) \( f(n) = n^2 \)
(d) \( f(n) = n^2 + n \)
(e) \( f(n) = n^3 \)
(f) \( f(n) = 2^n \)

Exercise 13.2 (Big-O, 2 + 1 marks)
Consider the Turing machine below. The input alphabet is \( \Sigma = \mathbb{N} = \{1, 2, 3, \ldots\} \). The operator \(|w|\) denotes the length of the string \( w \), the relation \(<\) is the smaller relation on the natural numbers.

\[
M = \text{"On input string } w\text{:}
\text{for } i = 1 \text{ to } |w|:
\text{for } j = |w| \text{ downto } i + 1:
\text{if } w_j < w_{j-1}:
\text{swap } w_j \text{ and } w_{j-1}:
\text{endif}
\text{endfor}
\text{endfor}
\]
Assume that the runtime of a swap and of a comparison of two natural numbers is constant.

(a) What is the smallest integer \( k \) such that the runtime of the Turing machine \( M \) is in \( O(|w|^k) \)? Justify your answer.
(b) What does \( M \) compute (i.e. what is written on the tape when \( M \) halts)?

Exercise 13.3 (Big-O, 2 + 3 marks)
Prove the following statements using the definition of Big-O:

(a) \( f(x) = 4x^3 + 2x - 4 \in O(x^3) \)
(b) \( g(n) = n \log n \not\in O(n) \)