Exercise 1.1 (Proof by induction, 2+3 marks)
Prove the following two statements by induction. Please make clear what is the base case, the induction hypothesis and the induction step.

(a) For $n \in \mathbb{N}^+$:
\[ \sum_{i=1}^{n}(2i - 1) = n^2 \]

(b) For $n \in \mathbb{N}^+$, the power set of the set $S_n = \{1, 2, \ldots, n\}$ contains $2^n$ elements.

Explanation: The power set $\mathcal{P}(S)$ of a set $S$ is the set of all subsets of $S$, e.g. $\mathcal{P}(\{1, 2\}) = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$.

Exercise 1.2 (Direct proof and proof by contradiction, 2+3 marks)
Consider the following definitions of graph, path, and tree.

- A graph $G = (V, E)$ is defined by a finite set of vertices (or nodes) $V$ together with a set of edges $E$ where each edge is a 2-element subset of $V$ (a subset $\{v_i, v_j\}$ in $E$ means that there is an edge between node $v_i$ and node $v_j$).
- A path is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. A simple path is a path that contains each vertex at most once.
- A tree is a connected graph that has no simple cycles. A graph is connected if each pair of distinct vertices can be connected through some path. A simple cycle in a graph is a path $\langle v_0, v_1, \ldots, v_n \rangle$ with $v_0 = v_n$, $n \geq 3$ and $v_i \neq v_j$ for all $0 \leq i < j < n$.

Consider the following statement:

Let $G = (V, E)$ be a graph with a vertex $v$ such that there is exactly one simple path from $v$ to each other vertex in $V$. Then $G$ is a tree.

In order to prove that $G$ is a tree, we have to prove that $G$ is connected and that $G$ does not contain simple cycles.

(a) Prove directly that $G$ is connected.

(b) Prove by contradiction that $G$ does not contain simple cycles.