Principles of AI Planning

9. Invariants

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Spurious formulae in regression planning

Example

Consider the goal formula

\[ A\text{-on-}B \land B\text{-on-}C \]

regressed with operator

\[ \langle A\text{-on-}C \land A\text{-clear} \land B\text{-clear}, A\text{-on-}B \land \neg B\text{-clear} \land C\text{-clear} \rangle \]

resulting in the new subgoal

\[ A\text{-on-}C \land A\text{-clear} \land B\text{-clear} \land B\text{-on-}C. \]

It is intuitively clear that no state satisfying this formula is reachable by any plan from a legal blocks world state.
Spurious formulae cause unnecessary search

- Goal formulae and formulae obtained by regressing them often represent some states that are not reachable from the initial state.
- If none of the states is reachable from the initial state, there are no plans reaching the formula.
- We would like to have reachable states only, if possible.
- The same problem shows up in satisfiability planning (discussed later in the course): partial valuations considered by satisfiability algorithms may represent unreachable states, and this may result in unnecessary search.
Restricting search to reachable sets

**Goal:** Restriction to states that are reachable.

**Problem:** Testing reachability is computationally as complex as testing whether a plan exists.

**Solution:** Use an *approximate* notion of reachability.

**Implementation:** Compute in polynomial time *formulae* that characterize a *superset* of the reachable states.
Invariants

Definition (invariant)
A formula $\varphi$ is an invariant of $\langle A, I, O, G \rangle$ if $s \models \varphi$ for every state $s$ reachable from $I$.

Example
The formula $\neg(A\text{-on-}B \land A\text{-on-}C)$ is an invariant in a well-formed blocks world task.

Remark
Invariants are usually proved inductively:
- Prove that $\varphi$ is true in the initial state.
- Prove that operator application preserves $\varphi$. 
Strongest invariants

**Definition (strongest invariant)**

An invariant $\varphi$ is **the strongest invariant** of $\langle A, I, O, G \rangle$ iff for any invariant $\psi$, $\varphi \models \psi$.

The strongest invariant **exactly characterizes** the set of all states that are reachable from the initial state:
For all states $s$, $s \models \varphi$ if and only if $s$ is reachable.

**Remark**

*There are infinitely many strongest invariants for any given planning task, but they are all logically equivalent.*

*(If $\varphi$ is a strongest invariant, then so is $\varphi \land \top$, $\varphi \lor \varphi$, ...)
Example: strongest invariant for blocks world

Example (blocks world)

Let $X$ be the set of blocks of a well-formed blocks world task $\Pi$, for example $X = \{A, B, C, D\}$.

The conjunction of the following formulae is the strongest invariant for $\Pi$:

For all $x \in X$: $\text{clear}(x) \leftrightarrow \bigwedge_{y \in X} \neg \text{on}(y, x)$

For all $x \in X$: $\text{ontable}(x) \leftrightarrow \bigwedge_{y \in X} \neg \text{on}(x, y)$

For all $x, y, z \in X$ with $y \neq z$: $\neg \text{on}(x, y) \lor \neg \text{on}(x, z)$

For all $x, y, z \in X$ with $y \neq z$: $\neg \text{on}(y, x) \lor \neg \text{on}(z, x)$

For all $n \geq 1$ and $x_1, \ldots, x_n \in X$:

$\neg (\text{on}(x_1, x_2) \land \text{on}(x_2, x_3) \land \cdots \land \text{on}(x_{n-1}, x_n) \land \text{on}(x_n, x_1))$
Theorem (strongest invariants vs. plan existence)

Let $\varphi$ be the strongest invariant for $\Pi = \langle A, I, O, G \rangle$.
Then $\Pi$ has a plan if and only if $G \land \varphi$ is satisfiable.

Proof.
Obvious.
Strongest invariants: complexity

**Theorem (complexity of computing strongest invariants)**

*Computing the strongest invariant $\varphi$ is PSPACE-hard.*

*Even deciding whether or not $\top$ is the strongest invariant is already PSPACE-hard.*

**Proof.**

By reduction from the plan existence problem.  
**Fact:** Testing plan existence for $\langle A, I, O, G \rangle$ is PSPACE-hard. (We’ll show this later in the course!)

Let $a' \notin A$ be a new state variable. Then a plan exists for $\Pi = \langle A, I, O, G \rangle$ iff $\top$ is the strongest invariant of the planning task $\Pi' = \langle A \cup \{a'\}, I \cup \{a' \mapsto 0\}, O \cup O', G \rangle$, where $O' = \{ \langle G, a' \land \bigwedge_{a \in A} a \rangle \} \cup \{ \langle a', \neg a \rangle \mid a \in A \cup \{a'\} \}$. 

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By reduction from the plan existence problem.

**Fact:** Testing plan existence for \( \langle A, I, O, G \rangle \) is PSPACE-hard. (We'll show this later in the course!)

Let \( a' \notin A \) be a new state variable. Then a plan exists for \( \Pi = \langle A, I, O, G \rangle \) iff \( \top \) is the strongest invariant of the planning task \( \Pi' = \langle A \cup \{a'\}, I \cup \{a' \mapsto 0\}, O \cup O', G \rangle \), where \( O' = \{ \langle G, a' \land \bigwedge_{a \in A} a \rangle \} \cup \{ \langle a', \neg a \rangle \mid a \in A \cup \{a'\} \} \).

...
Strongest invariants: complexity (ctd.)

Proof (ctd.)

($\Rightarrow$): If a plan exists for $\Pi$, then the same plan is applicable in $\Pi'$. We can thus reach a state satisfying $G$ in $\Pi'$.

From this state, we can reach any state $s$ by first applying $\langle G, a' \land \bigwedge_{a \in A} a \rangle$ and then applying the operators $\langle a', \neg a \rangle$ for each variable $a$ with $s(a) = 0$. (If $s(a') = 0$, the corresponding operator must be applied last.)

If all states are reachable in $\Pi'$, then $\top$ is the strongest invariant for $\Pi'$.

($\Leftarrow$) (by contraposition): If $\Pi$ is not solvable, then no state satisfying $G$ is reachable in $\Pi$. In that case, no state satisfying $G$ is reachable in $\Pi'$, and thus $a'$ cannot be made true in $\Pi'$. Thus, $\neg a'$ is an invariant in $\Pi'$ which is stronger than $\top$, so $\top$ is not the strongest invariant in $\Pi'$. 
Proof (ctd.)

$(\Rightarrow)$: If a plan exists for $\Pi$, then the same plan is applicable in $\Pi'$. We can thus reach a state satisfying $G$ in $\Pi'$. From this state, we can reach any state $s$ by first applying $\langle G, a' \land \bigwedge_{a \in A} a \rangle$ and then applying the operators $\langle a', \neg a \rangle$ for each variable $a$ with $s(a) = 0$. (If $s(a') = 0$, the corresponding operator must be applied last.)

If all states are reachable in $\Pi'$, then $\top$ is the strongest invariant for $\Pi'$.

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Strongest invariants: complexity (ctd.)

Proof (ctd.)

(⇒): If a plan exists for \( \Pi \), then the same plan is applicable in \( \Pi' \). We can thus reach a state satisfying \( G \) in \( \Pi' \).
From this state, we can reach any state \( s \) by first applying \( \langle G, a' \land \bigwedge_{a \in A} a \rangle \) and then applying the operators \( \langle a', \neg a \rangle \) for each variable \( a \) with \( s(a) = 0 \). (If \( s(a') = 0 \), the corresponding operator must be applied last.)
If all states are reachable in \( \Pi' \), then \( \top \) is the strongest invariant for \( \Pi' \).

(⇐) (by contraposition): If \( \Pi \) is not solvable, then no state satisfying \( G \) is reachable in \( \Pi \). In that case, no state satisfying \( G \) is reachable in \( \Pi' \), and thus \( a' \) cannot be made true in \( \Pi' \). Thus, \( \neg a' \) is an invariant in \( \Pi' \) which is stronger than \( \top \), so \( \top \) is not the strongest invariant in \( \Pi' \).
Strongest invariants: complexity (ctd.)

Proof (ctd.)

(⇒): If a plan exists for \( \Pi \), then the same plan is applicable in \( \Pi' \). We can thus reach a state satisfying \( G \) in \( \Pi' \). From this state, we can reach any state \( s \) by first applying \( \langle G, a' \land \bigwedge_{a \in A} a \rangle \) and then applying the operators \( \langle a', \neg a \rangle \) for each variable \( a \) with \( s(a) = 0 \). (If \( s(a') = 0 \), the corresponding operator must be applied last.) If all states are reachable in \( \Pi' \), then \( \top \) is the strongest invariant for \( \Pi' \).

(⇐) (by contraposition): If \( \Pi \) is not solvable, then no state satisfying \( G \) is reachable in \( \Pi \). In that case, no state satisfying \( G \) is reachable in \( \Pi' \), and thus \( a' \) cannot be made true in \( \Pi' \). Thus, \( \neg a' \) is an invariant in \( \Pi' \) which is stronger than \( \top \), so \( \top \) is not the strongest invariant in \( \Pi' \).
Invariant synthesis: example run

Compute sets $C_i$ of $n$-literal clauses characterizing (giving an upper bound!) the states that are reachable in up to $i$ steps.

Example

\[
C_0 = \{a, \neg b, c\} \sim \{101\}
\]
\[
C_1 = \{a \lor b, \neg a \lor \neg b, c\} \sim \{101, 011\}
\]
\[
C_2 = \{\neg a \lor \neg b, c\} \sim \{001, 011, 101\}
\]
\[
C_3 = \{\neg a \lor \neg b, c \lor a\} \sim \{001, 011, 100, 101\}
\]
\[
C_4 = \{\neg a \lor \neg b\} \sim \{000, 001, 010, 011, 100, 101\}
\]
\[
C_5 = \{\neg a \lor \neg b\} \sim \{000, 001, 010, 011, 100, 101\}
\]
\[
C_i = C_5 \text{ for all } i > 5
\]

\[\neg a \lor \neg b\] is the only invariant found.
Invariant synthesis algorithm (informally)

- Start with all 1-literal clauses true in the initial state.
- Repeatedly test every operator vs. every clause to check whether the clause can be shown to be true after applying the operator:
  - One of the literals in the clause is necessarily true: retain.
  - Otherwise, if the clause is too long: forget it.
  - Otherwise, replace the clause by new clauses obtained by adding literals that are now true.
- When all clauses are retained, stop: they are invariants.
Blocks world example

Example (blocks world)

Let $C_0 = \{A\text{-clear}, \neg B\text{-clear}, A\text{-on-B}, \neg B\text{-on-A}, \neg A\text{-on-T}, B\text{-on-T}\}$ and $o = \langle A\text{-clear} \land A\text{-on-B}, B\text{-clear} \land \neg A\text{-on-B} \land A\text{-on-T} \rangle$.

1. $C_0 \cup \{A\text{-clear} \land A\text{-on-B}\}$ is satisfiable: $o$ is applicable.

2. The 1-literal clauses $\neg B\text{-clear}$, $A\text{-on-B}$ and $\neg A\text{-on-T}$ become false when $o$ is applied.

3. They are not thrown away, though: they are replaced by weaker clauses.

4. Literals true after applying $o$ in state $s$ such that $s \models C_0$: $A\text{-clear}$, $B\text{-clear}$, $\neg A\text{-on-B}$, $\neg B\text{-on-A}$, $A\text{-on-T}$, $B\text{-on-T}$.

5. 2-literal clauses that are weaker than $\neg B\text{-clear}$ and now true are $\neg B\text{-clear} \lor A\text{-clear}$, $\neg B\text{-clear} \lor B\text{-clear}$, $\neg B\text{-clear} \lor \neg A\text{-on-B}$, $\neg B\text{-clear} \lor \neg B\text{-on-A}$, $\neg B\text{-clear} \lor A\text{-on-T}$, and $\neg B\text{-clear} \lor B\text{-on-T}$.
Similar 2-literal clauses are obtained from $A$-$on$-$B$ and from $\neg A$-$on$-$T$.

By eliminating logically equivalent ones, tautologies, and clauses that follow from those in $C_0$ not falsified we get

$$C_1 = \{ A$-clear, $\neg B$-on-$A$, $B$-on-$T$, $\neg B$-clear $\lor \neg A$-on-$B$, $\neg B$-clear $\lor A$-on-$T$, $A$-on-$B$ $\lor$ $B$-clear, $A$-on-$B$ $\lor$ $A$-on-$T$, $\neg A$-on-$T$ $\lor$ $B$-clear, $\neg A$-on-$T$ $\lor$ $\neg A$-on-$B$ \}$$

for distance 1 states.

Some clauses in $C_1$ can be refined further by checking other operators whose preconditions are consistent with $C_1$. With a bit more computation, $C_i$ settles to a set containing all invariants for two blocks.
Simple travel example

Example (simple travel)

Let $C_i = \{\neg AinRome \lor \neg AinParis, \\
\neg AinRome \lor \neg AinNYC, \\
\neg AinParis \lor \neg AinNYC\}$,
\[ o = \langle AinRome, AinParis \land \neg AinRome \rangle. \]

- Does $o$ preserve truth of $\neg AinParis \lor \neg AinNYC$?

- Because $o$ makes $\neg AinParis$ false, we must show that $\neg AinNYC$ is true after applying $o$.

- But $\neg AinNYC$ is not even mentioned in $o$!

- However, since $AinRome$ is the precondition of $o$ and $\neg AinRome \lor \neg AinNYC$ was true before applying $o$, we can infer that $\neg AinNYC$ was true before applying $o$.

- Since $o$ does not make $\neg AinNYC$ false, it is true also after applying $o$, and then so is $\neg AinParis \lor \neg AinNYC$. 

Invariant synthesis: function \textit{preserves-clause}

Test if an operator preserves a clause

\begin{verbatim}
def preserves-clause(l_1 \lor \cdots \lor l_n, C, o):
    for each \( l \in \{l_1, \ldots, l_n\} \):
        if not preserves-literal(C, o, \{l_1, \ldots, l_n\} \setminus \{l\}, l):
            return false
    return true
\end{verbatim}

Test if an operator preserves a literal

\begin{verbatim}
def preserves-literal(C, o, L', l):
    \langle c, e \rangle := o
    C_{l} := C \cup \{c\} \cup \{EPC_{l}(e)\}
    return \( C_{l} \) is unsatisfiable
    or \( C_{l} \models EPC_{l'}(e) \) for some \( l' \in L' \)
    or \( C_{l} \models l' \land \neg EPC_{l'}(e) \) for some \( l' \in L' \)
\end{verbatim}
Function *preserves-clause*: examples

Let $C = \{c \lor b\}$.

- \text{preserves-clause}(a \lor b, C, \langle\neg c, c \land d\rangle) \text{ returns true}
- \text{preserves-clause}(a \lor b, C, \langle\neg c, \neg a \land b\rangle) \text{ returns true}
- \text{preserves-clause}(a \lor b, C, \langle b, \neg a\rangle) \text{ returns true}
- \text{preserves-clause}(a \lor b, C, \langle\neg c, \neg a\rangle) \text{ returns true}
- \text{preserves-clause}(a \lor b, C, \langle c, \neg a\rangle) \text{ returns false}
Correctness of function \textit{preserves-clause}

Lemma (correctness of \textit{preserves-clause})

Let $C$ be a set of clauses, $\varphi = l_1 \lor \cdots \lor l_n$ a clause, and $o$ an operator.

If $\text{preserves-clause}(\varphi, C, o)$ returns true, then $\text{app}_o(s) \models \varphi$ for every state $s$ such that $s \models C \cup \{\varphi\}$ and $\text{app}_o(s)$ is defined.

(Proof omitted.)
Incompleteness of function \textit{preserves-clause}

\begin{itemize}
  \item Let \( o = \langle a, \neg b \land (c \triangleright d) \land (\neg c \triangleright e) \rangle \).
  \item \textit{preserves-clause}(b \lor d \lor e, \emptyset, o) \) returns \textbf{false} because the \textit{preserves-literal} check for \( l = b \) fails:
    \begin{itemize}
      \item Operator \( o \) can make \( b \) false.
      \item It is \textbf{not guaranteed} that \( d \) is true in the resulting state.
      \item It is \textbf{not guaranteed} that \( e \) is true in the resulting state.
    \end{itemize}
  \item However, \( d \lor e \) is true after applying \( o \), and hence \( b \lor d \lor e \) will be true as well.
\end{itemize}
Invariant synthesis: outline of main procedure

1. $C = \text{the set of 1-literal clauses true in the initial state.}$
2. For each operator $o$ and clause $\varphi \in C$, test if $\varphi$ remains true when $o$ is applied.
3. If not, remove $\varphi$, and if the number of literals in $\varphi$ is less than $n$, add clauses $\varphi \lor l$ for each literal $l$ which is guaranteed to be true after applying $o$.
4. Remove all dominated invariants.
5. Repeat from step 2 if $C$ has changed in the previous two steps.
6. Otherwise every clause in $C$ is an invariant.

For any fixed limit $n$ on the size of the clauses, the number of iterations is $O(m^n)$ (where $m = |A|$ is the number of state variables) and hence polynomial.
Invariant synthesis: the main procedure

```
def invariants(A, I, O, n):
    C := { a ∈ A | I ⊨ a } ∪ { ¬a | a ∈ A, I ⊭ a }
    repeat:
        C' := C
        for each l_1 ∨ · · · ∨ l_m ∈ C' and o = ⟨c, e⟩ ∈ O
            with preserves-clause(l_1 ∨ · · · ∨ l_m, C', o) = false:
                C := C \ {l_1 ∨ · · · ∨ l_m}
                if m < n:
                    for each literal l:
                        if C' ∪ {c} ⊨ EPC_l(e) ∨ (l ∧ ¬EPC_l̄(e)):
                            C := C ∪ {l_1 ∨ · · · ∨ l_m ∨ l}
                    C := { φ ∈ C | ¬∃φ' ∈ C : φ' ⊨ φ and φ' ⊬ φ }
        until C = C'
    return C
```
Invariant synthesis: correctness

**Theorem (correctness of invariants)**

The procedure \texttt{invariants}(A, I, O, n) returns a set \( C \) of clauses with at most \( n \) literals such that for any applicable operator sequence \( o_1, \ldots, o_m \in O \): \( \text{app}_{o_1\ldots o_m}(I) \models C \).

**Proof.**

\[ A \models I \models C: \]

- The initial state satisfies the initial set of 1-literal clauses.
- All modifications to the clause set only make it logically weaker (i.e., \( C' \models C \) after each iteration of the main loop.)
- Thus the initial state satisfies the resulting clause set \( C \) by induction over the number of iterations.

\[ \ldots \]
If \( s \models C \) and \( \text{app}_o(s) \) is defined, then \( \text{app}_o(s) \models C \).

- In the last iteration of the procedure, no formula is removed from \( C = C' \), and hence \( \text{preserves-clause}(\varphi, C, o) \) is true for all clauses \( \varphi \in C \) and operators \( o \in O \).
- By the lemma, this means that \( \text{app}_o(s) \models \varphi \) for every state \( s \) such that \( s \models C \) and \( \text{app}_o(s) \) is defined.
- Since this is true for all clauses \( \varphi \in C \), we get \( \text{app}_o(s) \models C \) for every state \( s \) such that \( s \models C \) and \( \text{app}_o(s) \) is defined.

From A and B, the theorem follows by induction over the length of the operator sequence.
Why is the strongest invariant not always found?

- The function *preserves-clause* is incomplete for general operators (but complete for STRIPS operators.) Making it complete makes it NP-hard.
- The strongest invariant may require arbitrarily long clauses, so the restriction to clauses of any fixed length makes it impossible to represent it.

**Example**

The acyclicity of the *on* relation in the blocks world needs clauses of length $n$ when there are $n$ blocks.

- Practical implementations of the algorithm use polynomial time approximations of the tests for satisfiability and $|=\ldots$.
**Invariant synthesis: example**

**Initial state:**  $I \models a \land \neg b \land \neg c$

**Operators:**  
- $o_1 = \langle a, \neg a \land b \rangle$
- $o_2 = \langle b, \neg b \land c \rangle$
- $o_3 = \langle c, \neg c \land a \rangle$

**Computation:** Find invariants with at most 2 literals:

\[
\begin{align*}
C_0 &= \{ a, \neg b, \neg c \} \\
C_1 &= \{ \neg c, a \lor b, \neg b \lor \neg a \} \\
C_2 &= \{ \neg b \lor \neg a, \neg c \lor \neg a, \neg c \lor \neg b \} \\
C_3 &= \{ \neg b \lor \neg a, \neg c \lor \neg a, \neg c \lor \neg b \} \\
C_i &= C_2 \text{ for all } i \geq 2
\end{align*}
\]
Example

Regression of \( \text{in}(A, \text{Freiburg}) \) by
\[
\langle \text{in}(A, \text{Strasbour}), \neg \text{in}(A, \text{Strasbour}) \land \text{in}(A, \text{Paris}) \rangle
\]
gives \( \text{in}(A, \text{Freiburg}) \land \text{in}(A, \text{Strasbour}) \)

No state satisfying \( \text{in}(A, \text{Freiburg}) \land \text{in}(A, \text{Strasbour}) \) makes sense if \( A \) denotes some usual physical object.
Problem: Regression produces sets $T$ of states such that
- some states in $T$ are unreachable from $I$, or even
- all states in $T$ are unreachable from $I$.

The first is not always a serious problem (but may worsen the quality of distance estimates, for example.)

Solution: Use invariants to avoid formulae that do not represent any reachable states.

1. Compute invariant $\varphi$.
2. Do only regression steps such that $\text{regr}_o(\psi) \land \varphi$ is satisfiable.
Exploiting invariants in satisfiability planning

- Invariants are very useful in the **planning as satisfiability** framework (SAT planning), where they help reduce the search space for the SAT solver.
- We will discuss SAT planning later in this course.
Invariants for problem reformulation: mutexes

Binary clause invariants are called *mutexes* because they state that certain variable assignments cannot be simultaneously true and are hence *mutually exclusive*.

**Example**

The invariant \( \neg A-on-B \lor \neg A-on-C \) states that *A-on-B* and *A-on-C* are mutex.

Often, a larger set of literals is mutually exclusive because every pair of them forms a mutex.

**Example**

In blocks world, *B-on-A*, *C-on-A*, *D-on-A* and *A-clear* are mutex.
Encoding mutex groups as finite-domain variables

Let $L = \{l_1, \ldots, l_n\}$ be mutually exclusive literals over $n$ different variables $A_L = \{a_1, \ldots, a_n\}$.

Then the planning task can be rephrased using a single finite-domain (i.e., non-binary) state variable $v_L$ with $n + 1$ possible values in place of the $n$ variables in $A_L$:

- $n$ of the possible values represent situations in which exactly one of the literals in $L$ is true.
- The remaining value represents situations in which none of the literals in $L$ is true.

Note: If we can prove that one of the literals in $L$ has to be true in each state, this additional value can be omitted.

In many cases, the reduction in the number of variables can dramatically improve performance of a planning algorithm.
Finite-domain state variables

**Definition (finite-domain state variable)**

A **finite-domain state variable** is a symbol $v$ with an associated **finite domain**, i.e., a non-empty finite set. We write $D_v$ for the domain of $v$.

**Example**

$v = \text{above-a}, D_{\text{above-a}} = \{b, c, d, \text{nothing}\}$

This state variable encodes the same information as the propositional variables $B\text{-on-A}, C\text{-on-A}, D\text{-on-A}$ and $A\text{-clear}$. 
Finite-domain states

Definition (finite-domain state)
Let $V$ be a finite set of finite-domain state variables.
A state over $V$ is an assignment $s : V \rightarrow \bigcup_{v \in V} D_v$ such that $s(v) \in D_v$ for all $v \in V$.

Example
$s = \{\text{above-a} \mapsto \text{nothing}, \text{above-b} \mapsto a, \text{above-c} \mapsto b,\text{below-a} \mapsto b, \text{below-b} \mapsto c, \text{below-c} \mapsto \text{table}\}$
Finite-domain formulae

Definition (finite-domain formulae)

Logical formulae over finite-domain state variables $V$ are defined as in the propositional case, except that instead of atomic formulae of the form $a \in A$, there are atomic formulae of the form $v = d$, where $v \in V$ and $d \in D_v$.

Example

The formulae $(above-a = nothing) \lor \neg(below-b = c)$ corresponds to the formula $A-clear \lor \neg B-on-C$. 
Finite-domain effects

Definition (finite-domain effects)
Effects over finite-domain state variables $V$ are defined as in the propositional case, except that instead of atomic effects of the form $a$ and $\neg a$ with $a \in A$, there are atomic effects of the form $v := d$, where $v \in V$ and $d \in D_v$.

Example
The effect
\[(\text{below-a} := \text{table}) \land ((\text{above-b} = a) \triangleright (\text{above-b} := \text{nothing}))\]
corresponds to the effect
\[
\text{A-on-T} \land \neg \text{A-on-B} \land \neg \text{A-on-C} \land \neg \text{A-on-D} \land (\text{A-on-B} \triangleright (\neg \text{A-on-B} \land \text{B-clear})).
\]
\[\Rightarrow \text{definition of finite-domain operators follows}\]
Planning tasks in finite-domain representation

Definition (planning task in finite-domain representation)

A deterministic planning task in finite-domain representation or FDR planning task is a 4-tuple $\Pi = \langle V, I, O, G \rangle$ where

- $V$ is a finite set of finite-domain state variables,
- $I$ is an initial state over $V$,
- $O$ is a finite set of finite-domain operators over $V$, and
- $G$ is a formula over $V$ describing the goal states.
Relationship to propositional planning tasks

**Definition (induced propositional planning task)**

Let $\Pi = \langle V, I, O, G \rangle$ be an FDR planning task. The **induced propositional planning task** $\Pi'$ is the (regular) planning task $\Pi' = \langle A', I', O', G' \rangle$, where

- $A' = \{(v, d) \mid v \in V, d \in D_v \}$
- $I'(v, d) = 1$ iff $I(v) = d$
- $O'$ and $G'$ are obtained from $O$ and $G$ by replacing
  - each atomic formula $v = d$ with the proposition $(v, d)$, and
  - each atomic effect $v := d$ with the effect $(v, d) \land \bigwedge_{d' \in D \setminus \{d\}} \neg(v, d').$

$\rightsquigarrow$ can define operator semantics, plans, relaxed planning graphs, ... for $\Pi$ in terms of its induced propositional planning task.
Definition (SAS$^+$ planning task)

An FDR planning task $\Pi = \langle V, I, O, G \rangle$ is called an SAS$^+$ planning task iff there are no conditional effects in $O$ and all operator preconditions in $O$ and the goal formula $G$ are conjunctions of atoms.

- analogue of STRIPS planning tasks for finite-domain representations
- induced propositional planning task of a SAS$^+$ planning task is STRIPS
- FDR tasks obtained by invariant-based reformulation of STRIPS planning task are SAS$^+$
DISCOPLAN (Gerevini & Schubert, 1998)

- many classes of invariants (not just mutexes), but not general clausal invariants
- *generate/test/repair* approach
  (similar to the algorithm presented here)
- limited to STRIPS
- works directly with *schematic operators*
- usually fast, but too expensive for some large tasks
TIM (Fox & Long, 1998)

- mutexes + some additional invariants
- **not a generate/test/repair approach**
  (or at least, not described as such)
- limited to STRIPS
- works directly with schematic operators
Literature on invariant synthesis (ctd.)

Edelkamp & Helmert's algorithm (1999)

- only mutexes
- specifically tailored towards FDR reformulation
- generate/test/repair approach
  (similar to the algorithm presented here)
- limited to STRIPS
- works directly with schematic operators
- fast, but limitations in PDDL support
  (even in addition to being STRIPS only)
Rintanen’s algorithm (2000)

- general clausal invariants
  - however, speed unclear for general invariants (beyond mutexes)
- generate/test/repair approach
- limited to STRIPS
- works with schematic operators

The algorithm presented in this section is essentially Rintanen’s algorithm, translated to non-schematic operators.
Bonet & Geffner’s algorithm (2001)

- mutexes only
- *generate/test* approach (without repair stage)
- limited to STRIPS
- works with *propositional representation* (not schematic)
- can be seen as simpler version of Rintanen’s algorithm
- quite expensive for very large planning tasks
- developed for additional pruning in *regression search*
Literature on invariant synthesis (ctd.)

Helmert’s algorithm (2009)

- only mutexes
- specifically tailored towards FDR reformulation
- generate/test/repair approach
  (similar to the algorithm presented here)
- not limited to STRIPS
- works directly with schematic operators
- fast
Invariants help make backward search and satisfiability planning more efficient and (in the case of mutexes) can be used for problem reformulation.

We gave an algorithm for computing a class of invariants.

1. Start with 1-literal clauses true in the initial state.
2. Repeatedly weaken clauses that could not be shown to be invariants.
3. Stop when all clauses are guaranteed to be invariants.

The algorithm runs in polynomial time if the satisfiability and logical consequence tests are approximated by a polynomial time algorithm and the size of the invariant clauses is bounded by a constant.