Exercise 9.1 (Specification morphisms)
Let $SP_1$, $SP_2$ be two specifications. Show that for any signature morphism $\sigma : \text{Sig}(SP_1) \to \text{Sig}(SP_2)$, the following are equivalent:

(a) $\sigma : SP_1 \to SP_2$ is a specification morphism
(b) $\text{Mod}(SP_2 \text{ hide } \sigma) \subseteq \text{Mod}(SP_1)$
(c) $\text{Mod}(SP_2) \subseteq \text{Mod}(SP_1 \text{ with } \sigma)$

Exercise 9.2 (Models of specifications)
Show that the following statements are not equivalent. Provide counterexamples for both implications.

(a) $\text{Mod}(SP_1) \subseteq \text{Mod}(SP_2 \text{ hide } \sigma)$
(b) $\text{Mod}(SP_1 \text{ with } \sigma) \subseteq \text{Mod}(SP_2)$

Exercise 9.3 (Algebraic laws for specifications)
Check which of the following algebraic laws hold:

(a) $SP$ and $SP \equiv SP$
(b) $SP_1$ and $SP_2 \equiv SP_2$ and $SP_1$
(c) $(SP \text{ with } \sigma_1)$ with $\sigma_2 \equiv SP$ with $\sigma_2 \circ \sigma_1$
(d) $(SP_1$ and $SP_2)$ with $\sigma \equiv (SP_1$ with $\sigma)$ and $(SP_2$ with $\sigma)$
(e) $(SP \text{ hide } \sigma_2)$ hide $\sigma_1 \equiv SP \text{ hide } \sigma_2 \circ \sigma_1$
(f) $(SP_1$ and $SP_2)$ hide $\sigma \equiv (SP_1$ hide $\sigma)$ and $(SP_2$ hide $\sigma)$
(g) $(SP \text{ with } \sigma)$ hide $\sigma \equiv SP$
(h) $(SP \text{ hide } \sigma)$ with $\sigma \equiv SP$

Exercise 9.4 (Christmas bonus problem: existence of Santa Clause I)
Explain what is wrong with the following proof of the existence of Santa Clause.

Recall the $\exists$-introduction rule:

\[
\frac{\varphi(t)}{\exists x : s. \varphi(x)}
\]

Theorem. Santa Clause exists.
Proof. Assume to the contrary that Santa Clause does not exist. By $\exists$-introduction, there exists something that does not exist. This is a contradiction. Hence, the assumption that Santa Clause does not exist must be wrong. Thus, Santa Clause exists. \qed
**Exercise 9.5** (Christmas bonus problem: existence of Santa Clause II)

So, the existence proof in the last exercise was flawed. But I have another proof. What about this one?

Let $c$ be the set \( \{ x | \text{if } x \in x, \text{ then Santa Clause exists} \} \).
Now, assume that $c \in c$. We know that $c \in c$ iff “if $c \in c$, then Santa Clause exists”. Therefore, we may assert that “if $c \in c$, then Santa Clause exists”. Thus by modus ponens, Santa Clause exists.
However, I have just proven that “if $c \in c$, then Santa Clause exists”. Thus, $c \in c$, and therefore Santa Clause exists.

\( \square \)

**Exercise 9.6** (Christmas bonus problem: all reindeers have the same color)

So I could not convince you that Santa Clause exists. Can I convince you that all reindeers have the same color? The following proof should assure you of that claim, shouldn’t it?

*Theorem.* Any number of reindeers have the same color.

*Proof.* By induction.

Basis: one reindeer has the same color (obviously!).

Inductive step: suppose that any collection of $n$ reindeers has the same color.

We need to show that $n + 1$ reindeers have the same color, too. By induction hypothesis, the first $n$ reindeers have the same color. Take out the last reindeer of these and replace it with the $n+1$st. Again by induction hypothesis, these have the same color. Hence, all $n + 1$ reindeers have the same color.

\( \square \)

The exercise sheets may and should be worked on in groups of two (2) students. Please write both names on your solution.