Principles of AI Planning

Computational complexity

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We have seen that planning in transition systems can be done in time polynomial in the size of the transition system. This appears not to be true for planning in succinct transition systems (= planning tasks).

1. What is the precise computational complexity of the planning problem?
2. How does the computational complexity vary with the expressiveness of the planning language?
3. What is the computational complexity of planning in a particular domain (e.g. blocks world)?
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Computational Complexity

Motivation

We have seen that planning in transition systems can be done in time polynomial in the size of the transition system.

This appears not to be true for planning in succinct transition systems (= planning tasks).

1. What is the precise computational complexity of the planning problem?
2. How does the computational complexity vary with the expressiveness of the planning language?
3. What is the computational complexity of planning in a particular domain (e.g. blocks world)?
Why Computational Complexity?

- understand the problem
- know what is not possible
- find interesting subproblems that are easier to solve
- distinguish essential features from syntactic sugar
### Deterministic planning: NP-hardness

**Definition**

The decision problem SAT: test whether a given propositional formula $\phi$ is satisfiable.

**Reduction from SAT to deterministic planning**

\[
A = \text{the set of propositional variables occurring in } \phi \\
I = \text{any state, e.g. all state variables have value 0} \\
O = (\{T\} \times A) \cup (\{\langle T, \neg a \rangle | a \in A\})
\]

There is a plan for $\langle A, I, O, \phi \rangle$ if and only if $\phi$ is satisfiable.
Deterministic planning: NP-hardness

Because there is a polynomial-time translation from SAT into deterministic planning, and SAT is an NP-complete problem, there is a polynomial time translation from every decision problem in NP into deterministic planning. Hence the problem is NP-hard.

1. Does NP-hardness depend on having Boolean formulae as preconditions?
2. Does deterministic planning have the power of NP, or is it still more powerful?
3. We show that it is more powerful: The decision problem of testing whether a plan exists is PSPACE-complete.
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Definition: Alternating Turing Machine

Alternating Turing Machine (ATM) \( \langle \Sigma, \Box, Q, q_0, l, \delta \rangle \):

1. input alphabet \( \Sigma \) and blank symbol \( \Box \notin \Sigma \)
   - alphabets always non-empty and finite
   - tape alphabet \( \Sigma_{\Box} = \Sigma \cup \{\Box\} \)

2. finite set \( Q \) of internal states with initial state \( q_0 \in Q \)

3. state labeling \( l : Q \rightarrow \{Y, N, \exists, \forall\} \)
   - accepting, rejecting, existential, universal states
   - terminal states \( Q_\star = Q_Y \cup Q_N \)
   - nonterminal states \( Q' = Q_\exists \cup Q_\forall \)

4. transition relation \( \delta \subseteq (Q' \times \Sigma_{\Box}) \times (Q \times \Sigma_{\Box} \times \{-1, +1\}) \)
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(Non-) Deterministic Turing Machines

Definition: Non-deterministic Turing Machine

A non-deterministic Turing Machine (NTM) is an ATM where all nonterminal states are existential.
- no universal states

Definition: Deterministic Turing Machine

A deterministic Turing Machine (DTM) is an NTM where the transition relation is functional.
- for all \((q, a) \in Q' \times \Sigma\), there is exactly one triple \((q', a', \Delta)\) with \(((q, a), (q', a', \Delta)) \in \delta\)
- notation: \(\delta(q, a) = (q', a', \Delta)\)
### (Non-) Deterministic Turing Machines

#### Definition: Non-deterministic Turing Machine

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- notation: \(\delta(q, a) = (q', a', \Delta)\)
Let $M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle$ be an ATM.

**Definition: Configuration**

A **configuration** of $M$ is a triple $(w, q, x) \in \Sigma^* \times Q \times \Sigma^+$.  

- $w$: tape contents before tape head  
- $q$: current state  
- $x$: tape contents after and including tape head
Turing Machine Transitions

Let \( M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle \) be an ATM.

**Definition: Yields relation**

A configuration \( c \) of \( M \) yields a configuration \( c' \) of \( M \), in symbols \( c \vdash c' \), as defined by the following rules, where \( a, a', b \in \Sigma \square, w, x \in \Sigma^*, q, q' \in Q \) and \(((q, a), (q', a', \Delta)) \in \delta:\)

\[
(w, q, ax) \vdash (wa', q', x) \quad \text{if } \Delta = +1, |x| \geq 1
\]
\[
(w, q, a) \vdash (wa', q', \square) \quad \text{if } \Delta = +1
\]
\[
(wb, q, ax) \vdash (w, q', ba'x) \quad \text{if } \Delta = -1
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(\epsilon, q, ax) \vdash (\epsilon, q', \square a'x) \quad \text{if } \Delta = -1
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Let $M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle$ be an ATM.

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A configuration $c$ of $M$ **yields** a configuration $c'$ of $M$, in symbols $c \vdash c'$, as defined by the following rules, where $a, a' \in \Sigma \square$, $w, x \in \Sigma^*$, $q, q' \in Q$ and $((q, a), (q', a', \Delta)) \in \delta$:

- $(w, q, ax) \vdash (wa', q', x)$ if $\Delta = +1, |x| \geq 1$
- $(w, q, a) \vdash (wa', q', \square)$ if $\Delta = +1$
- $(wb, q, ax) \vdash (w, q', bab'x)$ if $\Delta = -1$
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Acceptance (Time)

Let $M = \langle \Sigma, \Box, Q, q_0, l, \delta \rangle$ be an ATM.

**Definition: Acceptance (time)**

Let $c = (w, q, x)$ be a configuration of $M$.

- $M$ accepts $c = (w, q, x)$ with $q \in Q_\forall$ in time $n$ for all $n \in \mathbb{N}_0$.
- $M$ accepts $c = (w, q, x)$ with $q \in Q_\exists$ in time $n$ iff $M$ accepts some $c'$ with $c \vdash c'$ in time $n - 1$.
- $M$ accepts $c = (w, q, x)$ with $q \in Q_\forall$ in time $n$ iff $M$ accepts all $c'$ with $c \vdash c'$ in time $n - 1$. 
Let $M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle$ be an ATM.

**Definition: Acceptance (time)**

Let $c = (w, q, x)$ be a configuration of $M$.

- $M$ accepts $c = (w, q, x)$ with $q \in Q_Y$ in time $n$ for all $n \in \mathbb{N}_0$.
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Let $M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle$ be an ATM.

**Definition: Acceptance (space)**

Let $c = (w, q, x)$ be a configuration of $M$.

- $M$ accepts $c = (w, q, x)$ with $q \in Q_Y$ in space $n$ iff $|w| + |x| \leq n$.

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- **$M$ accepts** $c = (w, q, x)$ with $q \in Q_A$ in space $n$ iff $M$ accepts all $c'$ with $c \vdash c'$ in space $n$. 

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Accepting Words and Languages

Let \( M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle \) be an ATM.

**Definition: Accepting words**

\( M \) accepts the word \( w \in \Sigma^* \) in time (space) \( n \in \mathbb{N}_0 \) iff \( M \) accepts \( (\epsilon, q_0, w) \) in time (space) \( n \).

- Special case: \( M \) accepts \( \epsilon \) in time (space) \( n \in \mathbb{N}_0 \) iff \( M \) accepts \( (\epsilon, q_0, \square) \) in time (space) \( n \).

**Definition: Accepting languages**

Let \( f : \mathbb{N}_0 \to \mathbb{N}_0 \).

\( M \) accepts the language \( L \subseteq \Sigma^* \) in time (space) \( f \) iff \( M \) accepts each word \( w \in L \) in time (space) \( f(|w|) \), and \( M \) does not accept any word \( w \notin L \).
Let $M = \langle \Sigma, \square, Q, q_0, l, \delta \rangle$ be an ATM.

**Definition: Accepting words**

$M$ accepts the word $w \in \Sigma^*$ in time (space) $n \in \mathbb{N}_0$ iff $M$ accepts $(\epsilon, q_0, w)$ in time (space) $n$.

- **Special case:** $M$ accepts $\epsilon$ in time (space) $n \in \mathbb{N}_0$ iff $M$ accepts $(\epsilon, q_0, \square)$ in time (space) $n$.

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Let $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

$M$ accepts the language $L \subseteq \Sigma^*$ in time (space) $f$ iff $M$ accepts each word $w \in L$ in time (space) $f(|w|)$, and $M$ does not accept any word $w \notin L$. 
Definition: DTIME, NTIME, ATIME

Let $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

- Complexity class $\text{DTIME}(f)$ contains all languages accepted in time $f$ by some DTM.
- Complexity class $\text{NTIME}(f)$ contains all languages accepted in time $f$ by some NTM.
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Polynomial Complexity Classes

Let $\mathcal{P}$ be the set of polynomials $p : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

**Definition: P, NP, ...**

\[
\begin{align*}
P &= \bigcup_{p \in \mathcal{P}} \text{DTIME}(p) \\
\text{NP} &= \bigcup_{p \in \mathcal{P}} \text{NTIME}(p) \\
\text{AP} &= \bigcup_{p \in \mathcal{P}} \text{ATIME}(p) \\
\text{PSPACE} &= \bigcup_{p \in \mathcal{P}} \text{DSPACE}(p) \\
\text{NPSPACE} &= \bigcup_{p \in \mathcal{P}} \text{NSPACE}(p) \\
\text{APSPACE} &= \bigcup_{p \in \mathcal{P}} \text{ASPACE}(p)
\end{align*}
\]
Let $\mathcal{P}$ be the set of polynomials $p : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

**Definition: EXP, NEXP, ...**

<table>
<thead>
<tr>
<th>Class</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXP</td>
<td>$\bigcup_{p \in \mathcal{P}} \text{DTIME}(2^p)$</td>
</tr>
<tr>
<td>NEXP</td>
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</tr>
<tr>
<td>AEXP</td>
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</tr>
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Let $\mathcal{P}$ be the set of polynomials $p : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

**Definition: 2-EXP, ...**

$$2\text{-EXP} = \bigcup_{p \in \mathcal{P}} \text{DTIME}(2^{2^p})$$

...
### Standard Complexity Classes Relationships

<table>
<thead>
<tr>
<th>Complexity Class 1</th>
<th>Complexity Class 2</th>
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<tbody>
<tr>
<td>P</td>
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</tr>
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</table>

Theorem:

- \( P \subseteq NP \subseteq AP \)
- \( PSPACE \subseteq NPSPACE \subseteq APSPACE \)
- \( EXP \subseteq NEXP \subseteq AEXP \)
- \( EXPSPACE \subseteq NEXPSPACE \subseteq AEXPSPACE \)
- \( 2-EXP \subseteq \ldots \)
The Power of Nondeterministic Space

Theorem (Savitch 1970)

\[
\text{NSPACE}(f) \subseteq \text{DSPACE}(f^2), \text{ and thus:}
\]

\[
\text{PSPACE} = \text{NPSPACE}
\]

\[
\text{EXPSPACE} = \text{NEXPSPACE}
\]
The Power of Alternation (Chandra et al. 1981)

\[ \text{AP} = \text{PSPACE} \]
\[ \text{APSPACE} = \text{EXP} \]
\[ \text{AEXP} = \text{EXPSPACE} \]
\[ \text{AEXPSPACE} = 2\text{-EXP} \]
The Hierarchy of complexity classes

2-EXPSPACE = 2-NEXPSPACE

2-NEXP

2-EXP = AEXPSPACE

EXPSPACE = NEXPSPACE = AEXP

NEXP

EXP = APSPACE

PSPACE = NPSPACE = AP

NP

P
The Hierarchy of complexity classes

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The Planning Problem

**PlanEx (Plan Existence)**

**Given:** Planning task \( \langle A, I, O, G \rangle \)

**Question:** Is there a plan for \( \langle A, I, O, G \rangle \)?

**PlanLen (Bounded Plan Existence)**

**Given:** Planning task \( \langle A, I, O, G \rangle \), bound \( K \in \mathbb{N}_0 \)

**Question:** Is there a plan for \( \langle A, I, O, G \rangle \) of length at most \( K \)?
Plan Existence vs. Bounded Plan Existence

**PlanEx \leq_P PlanLen**

A planning task with \( n \) state variables has a plan iff it has a plan of length at most \( 2^n - 1 \).

\( \sim \) polynomial reduction
**PlanLen ∈ PSPACE**

Show **PlanLen ∈ NPSPACE** and use Savitch’s theorem.

Nondeterministic algorithm:

```python
def plan(⟨A, I, O, G⟩, K):
    s := I
    k := K
    repeat until s ⊨ G:
        guess o ∈ O
        reject if o not applicable in s
        set s := app_o(s)
        reject if k = 0
        set k := k − 1
    accept
```
Idea: **generic reduction**

- For a fixed polynomial $p$, given DTM $M$ and input $w$, generate planning task which is solvable iff $M$ accepts $w$ in space $p(|w|)$

- For simplicity, restrict to TMs which never move to the left of the initial head position (no loss of generality)
Hardness for PSPACE

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Reduction: State Variables

Let $p$ be the space-bound polynomial.

Given DTM $\langle \Sigma, \square, Q, q_0, l, \delta \rangle$ and input $w_1 \ldots w_n$, define relevant tape positions $X = \{1, \ldots, p(n)\}$.

State variables

- $\text{state}_q$ for all $q \in Q$
- $\text{head}_i$ for all $i \in X \cup \{0, p(n) + 1\}$
- $\text{content}_{i,a}$ for all $i \in X$, $a \in \Sigma_{\square}$
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Reduction: Initial State

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Operators

One operator for each transition rule \( \delta(q, a) = (q', a', \Delta) \) and each cell position \( i \in X \):

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- **effect**: \( \neg\text{state}_q \land \neg\text{head}_i \land \neg\text{content}_{i,a} \land \text{state}_{q'} \land \text{head}_{i+\Delta} \land \text{content}_{i,a'} \)
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Theorem (PSPACE-completeness (Bylander))

PlanEx and PlanLen are PSPACE-complete even if the planning task is given in STRIPS form (preconditions and goals are conjunctions of literals and no conditional effects).

Proof.

Hardness and membership for the general formalism follows from the above. Hardness holds for STRIPS as well because of the style of the reduction: only simple preconditions and no conditional effects. The only problem is the disjunction in the goal formula. This can be eliminated by transforming the TM beforehand, though.
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First-Order Tasks

- we considered **propositional** state variables (0-ary predicates) and **grounded** operators (0-ary schematic operators)
- reasonable: most planning algorithms work on grounded representations
- predicate arity is typically small (a constant?)

How do the complexity results change if we introduce first-order predicates and schematic operators?
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- Reasonable: most planning algorithms work on grounded representations.
- Predicate arity is typically small (a constant?)

How do the complexity results change if we introduce first-order predicates and schematic operators?
Membership in EXPSPACE

**PlanEx**, **PlanLen** ∈ EXPSPACE

- input size $n$
- $\sim$ at most $2^n$ grounded state variables
- $\sim$ at most $2^n$ grounded operators
- can ground the task in exponential time, then use the earlier PSPACE algorithms
Idea: Adapt the earlier reduction from PLANEx to encode Turing Machine contents more succinctly. Assume relevant tape positions are now $X = \{1, \ldots, 2^n\}$. We need to encode the computation as a planning task in polynomial time!

**Objects**

- $0, 1$

**Predicates**

- $\text{state}_q()$ for all $q \in Q$
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Reduction: Example Operator

**Operator example**

Schematic operator for transition rule \( \delta(q, a) = (q', a', +1) \)

- **parameters:** \(?b_1, \ldots, ?b_n\)
- **precondition:**
  - \(\text{state}_q\)
  - \(\land \text{head}(?b_1, \ldots, ?b_n)\)
  - \(\land \text{content}_a(?b_1, \ldots, ?b_n)\)
- **effect:**
  - \(\neg\text{state}_q\)
  - \(\land \neg\text{head}(?b_1, \ldots, ?b_n)\)
  - \(\land \neg\text{content}_a(?b_1, \ldots, ?b_n)\)
  - \(\land \text{state}_{q'}\)
  - \(\land \text{advance-head}\)
  - \(\land \text{content}_{a'}(?b_1, \ldots, ?b_n)\)
Reduction: Example Operator (continued)

**Operator example (ctd.)**

\[
\begin{align*}
\text{advance-head} &= ((?b_n = 0) \\
& \quad \triangleright \text{head}(?b_1, \ldots, ?b_{n-1}, 1)) \\
& \quad \land ((?b_{n-1} = 0 \land ?b_n = 1) \\
& \quad \quad \triangleright \text{head}(?b_1, \ldots, ?b_{n-2}, 1, 0)) \\
& \quad \land ((?b_{n-2} = 0 \land ?b_{n-1} = 1 \land ?b_n = 1) \\
& \quad \quad \triangleright \text{head}(?b_1, \ldots, ?b_{n-3}, 1, 0, 0)) \\
& \quad \land \ldots \\
& \quad \land ((?b_1 = 0 \land ?b_2 = 1 \land \cdots \land ?b_n = 1) \\
& \quad \quad \triangleright \text{head}(1, 0, \ldots, 0))
\end{align*}
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Plan Existence vs. Bounded Plan Existence

- Our earlier reduction from $\text{PlanEx}$ to $\text{PlanLen}$ no longer works: the shortest plan can have length doubly exponentially in the input size, so that the bound cannot be written down in polynomial time.

- Indeed, $\text{PlanLen}$ is actually easier than $\text{PlanEx}$ for this planning formalism (NEXP-complete).
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- Indeed, PlanLen is actually easier than PlanEx for this planning formalism (NEXP-complete).
If we allow in addition function terms with arity $> 0$, then planning becomes undecidable.

The state space is infinite: $s(0), s(s(0)), s(s(s(0))), \ldots$

We can use function terms to describe (the index of) tape cells of a Turing machine.

We can use operators to describe the Turing machine control.

The existence of a plan is then equivalent to the existence of a successful computation on the Turing machine.

PlanEx for planning tasks with function terms can be used to decide the Halting problem.

**Theorem**

PlanEx for planning tasks with function terms is undecidable.
Planning (and its complexity) for particular domains is interesting, since we want to **judge** planning benchmarks. …and perhaps want to go for domain-dependent planning. Consider **fixed domains** and determine complexity.
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A Concrete Domain: Logistics

There are several cities, each containing several locations, some of which are airports. There are also trucks, which drive within a single city, and airplanes, which can fly between airports. The goal is to get some packages from various locations to various new locations [McDermott, 1998].
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Plan Existence for Logistics

Theorem

PLANEX for Logistics can be decided in polynomial time.

Proof.

Consider the subgraphs formed by the connected airport networks (for planes) and city networks (for trucks). If at least one vehicle (truck or plane) is in one of the subgraphs, all nodes in the subgraph are internally reachable, otherwise only the externally connected nodes can be reached. Check for each package delivery, whether there are connected subgraphs such that the package can pass through the subgraphs to the target node. This is a simple reachability test, which can be done in poly. time.
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Definition (Feedback Vertex Set)

- Given: a directed graph $G = (V, A)$ and a natural number $k$
- Question: Does there exist a subset $V' \subseteq V$ with $|V'| \leq k$ such that removing $V'$ results in an acyclic graph?

This problem is NP-complete and can be used to prove the following result:

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PLANLEN for Logistics is NP-complete, even if there is only one complete city graph and one truck in this graph.

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Membership follows because there is an obvious polynomial upper bound of moves for all solvable instances. Hardness is shown using a reduction from FVS.
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Proof. (continued)

Let \( G = (V, A) \) be a directed graph and \( k \) a natural number. Then \( G \) contains a FVS of size \( k \) iff the logistics problem constructed below has a plan of length at most \( 3|V| + 2|A| + k \).

Construct a Logistics task with just one truck and one city network which is a complete graph containing \( V \) and an extra node \( v_0 \), where the truck starts. The truck has to deliver one package from \( v_0 \) to each other location and one package from \( u \) to \( v \) for each \( (u, v) \in A \).
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![Diagram of graphs](image)
PLANLEN for Logistics: NP-hardness contd. (2)

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Conversely, at least $3|V| + 2|A|$ actions are needed. If a plan contains not more than $3|V| + 2|A| + k$, then no more than $k$ nodes are visited twice. These nodes form a FVS of size $k$.

Note: It is not route planning that makes the task difficult, but the interaction of sub-goals!
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Other Domains

- Generalizations of all the domains that have been used at the international planning competition have been analyzed.
- Many show a similar behavior as Logistics: PLANEX is in P, PLANLEN is NP-complete, e.g., Blocks world.
- Some are already NP-complete for PLANEX, e.g. Freecell.
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Summary

- Planning using general **first-order terms** is **undecidable**.
- Planning using a function free language is **EXPSPACE-complete**.
- Planning with a propositional language (no schema variables) is **PSPACE-complete**.
- If we consider only “short” plans, the complexity comes down to **NP-completeness**.
- **Domain-dependent** planning can be easier.
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