

Principles of AI Planning

December 8th, 2006 — Planning with binary decision diagrams

Binary Decision Diagrams

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BDD operations

- Ideas
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- Derived operations

Planning with BDDs

- Main algorithm
- The apply function
- Remarks

Principles of AI Planning

Planning with binary decision diagrams

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Dealing with large state spaces

- ▶ One way to explore very large state spaces is to use **selective** exploration methods (such as heuristic search) that only explore a fraction of states.
- ▶ Another method is to **concisely represent** large sets of states and deal with large state sets at the same time.

Breadth-first search with progression and state sets

Progression breadth-first search

```
def bfs-progression(A, I, O, G):  
    goal := formula-to-set(G)  
    reached := {I}  
    loop:  
        if reached ∩ goal ≠ ∅:  
            return solution found  
        new-reached := reached ∪ apply(reached, O)  
        if new-reached = reached:  
            return no solution exists  
        reached := new-reached
```

⇝ If we can implement operations **formula-to-set**, **{I}**, **∩**, **≠ ∅**, **∪**, **apply** and **=** efficiently, this is a reasonable algorithm.

Formulae to represent state sets

- ▶ We have previously considered **boolean formulae** as a means of representing set of states.
- ▶ Compared to **explicit representations** of state sets, boolean formulae have very nice performance characteristics.

Note: In the following, we assume that formulae are implemented as **trees**, not **strings**, so that we can e.g. compute $\chi \wedge \psi$ from χ and ψ in **constant time**.

Performance characteristics

Explicit representations vs. formulae

Let k be the **number of state variables**, $|S|$ the **number of states** in S and $\|S\|$ the **size of the representation** of S .

	Sorted vector	Hash table	Formula
$s \in S?$	$O(k \log S)$	$O(k)$	$O(\ S\)$
$S := S \cup \{s\}$	$O(k \log S + S)$	$O(k)$	$O(k)$
$S := S \setminus \{s\}$	$O(k \log S + S)$	$O(k)$	$O(k)$
$S \cup S'$	$O(k S + k S')$	$O(k S + k S')$	$O(1)$
$S \cap S'$	$O(k S + k S')$	$O(k S + k S')$	$O(1)$
$S \setminus S'$	$O(k S + k S')$	$O(k S + k S')$	$O(1)$
\bar{S}	$O(k2^k)$	$O(k2^k)$	$O(1)$
$\{s \mid s(a) = 1\}$	$O(k2^k)$	$O(k2^k)$	$O(1)$
$S = \emptyset?$	$O(1)$	$O(1)$	co-NP-complete
$S = S'?$	$O(k S)$	$O(k S)$	co-NP-complete
$ S $	$O(1)$	$O(1)$	#P-complete

Which operations are important?

- ▶ **Explicit representations** such as hash tables are not suitable because their size grows linearly with the number of represented states.
- ▶ **Formulae** are very efficient for some operations, but not very well suited for other important operations needed by the progression algorithm.
 - ▶ Examples: $S \neq \emptyset?$, $S = S'?$
- ▶ One of the sources of difficulty is that formulae allow **many different representations** for a given set.
 - ▶ For example, all unsatisfiable formulae represent \emptyset .

This makes equality tests expensive.

\leadsto We are interested in **canonical representations**, i.e. representations for which there is only **one possible representation** for every state set.

Binary decision diagrams (BDDs) are an example of an efficient canonical representation.

Performance characteristics

Formulae vs. BDDs

Let k be the **number of state variables**, $|S|$ the **number of states** in S and $\|S\|$ the **size of the representation** of S .

	Formula	BDD
$s \in S?$	$O(\ S\)$	$O(k)$
$S := S \cup \{s\}$	$O(k)$	$O(k)$
$S := S \setminus \{s\}$	$O(k)$	$O(k)$
$S \cup S'$	$O(1)$	$O(\ S\ \ S'\)$
$S \cap S'$	$O(1)$	$O(\ S\ \ S'\)$
$S \setminus S'$	$O(1)$	$O(\ S\ \ S'\)$
\bar{S}	$O(1)$	$O(\ S\)$
$\{s \mid s(a) = 1\}$	$O(1)$	$O(1)$
$S = \emptyset?$	co-NP-complete	$O(1)$
$S = S'?$	co-NP-complete	$O(1)$
$ S $	#P-complete	$O(\ S\)$

Remark: Optimizations allow BDDs with complementation (\bar{S}) in constant time, but we will not discuss this here.

Binary decision diagrams

Definition

Definition (BDD)

Let A be a set of propositional variables.

A **binary decision diagram (BDD)** over A is a directed acyclic graph with labeled arcs and labeled vertices satisfying the following conditions:

- ▶ There is exactly one node without incoming arcs.
- ▶ All sinks (nodes without outgoing arcs) are labeled **0** or **1**.
- ▶ All other nodes are labeled with a variable $a \in A$ and have exactly two outgoing arcs, labeled **0** and **1**.

Binary decision diagrams

Terminology

BDD terminology

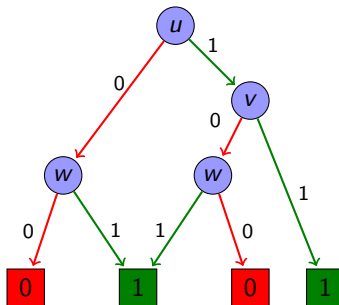
- ▶ The node without incoming arcs is called the **root**.
- ▶ The labeling variable of an internal node is called the **decision variable** of the node.
- ▶ The nodes reached from node n via the arc labeled $i \in \{0, 1\}$ is called the **i -successor** of n .
- ▶ The BDDs which only consist of a single sink are called the **zero BDD** and **one BDD**, respectively.

Observation: If B is a BDD and n is a node of B , then the subgraph induced by all nodes reachable from n is also a BDD.

- ▶ This BDD is called the **BDD rooted at n** .

BDD example

Possible BDD for $(u \wedge v) \vee w$



BDD semantics

Testing whether a BDD includes a valuation

def `bdd-includes(B : BDD, v : valuation)`:

Set n to the root of B .

while n is not a sink:

Set a to the decision variable of n .

Set n to the $v(a)$ -successor of n .

return true if n is labeled 1, false if it is labeled 0.

Definition (set represented by a BDD)

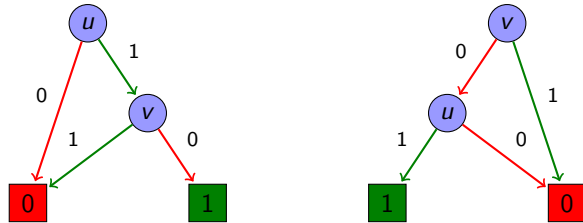
Let B be a BDD over variables A . The **set represented by B** , in symbols $r(B)$ consists of all valuations $v : A \rightarrow \{0, 1\}$ for which `bdd-includes(B, v)` returns true.

Ordered BDDs

Motivation

In general, BDDs are not a canonical representation for sets of valuations. Here is a simple counter-example ($A = \{u, v\}$):

BDDs for $u \wedge \neg v$ with different variable order



Both BDDs represent the same state set, namely the singleton set $\{u \mapsto 1, v \mapsto 0\}$.

Ordered BDDs

Definition

- ▶ As a first step towards a canonical representation, we will in the following assume that the set of variables A is **totally ordered** by some ordering \prec .
- ▶ In particular, we will only use variables v_1, v_2, v_3, \dots and assume the ordering $v_i \prec v_j$ iff $i < j$.

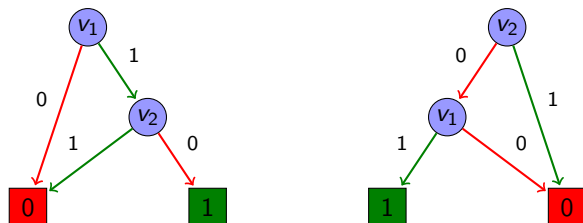
Definition (ordered BDD)

A BDD is **ordered** iff for each arc from an internal node with decision variable u to an internal node with decision variable v , we have $u \prec v$.

Ordered BDDs

Example

Ordered and unordered BDD

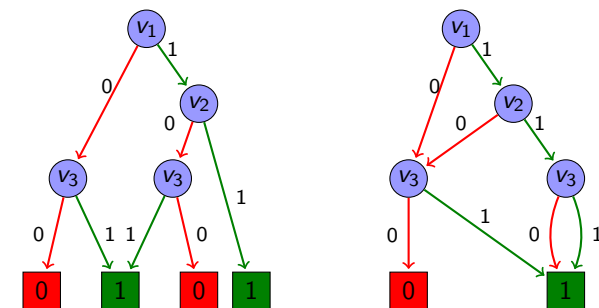


The left BDD is ordered, the right one is not.

Reduced ordered BDDs

Are ordered BDDs canonical?

Two equivalent BDDs that can be reduced



- ▶ Ordered BDDs are not canonical: Both ordered BDDs represent the same set.
- ▶ However, ordered BDDs can easily be **made** canonical.

Reduced ordered BDDs

Reductions

There are two important operations on BDDs that do not change the set represented by it:

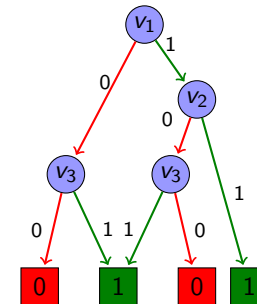
Definition (Isomorphism reduction)

If the BDDs rooted at two different nodes n and n' are **isomorphic**, then all incoming arcs of n' can be redirected to n , and all parts of the BDD no longer reachable from the root removed.

Reduced ordered BDDs

Reductions

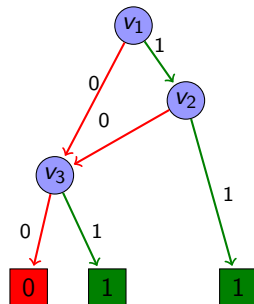
Isomorphism reduction



Reduced ordered BDDs

Reductions

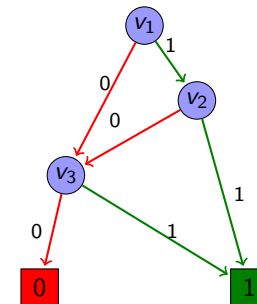
Isomorphism reduction



Reduced ordered BDDs

Reductions

Isomorphism reduction



Reduced ordered BDDs

Reductions

There are two important operations on BDDs that do not change the set represented by it:

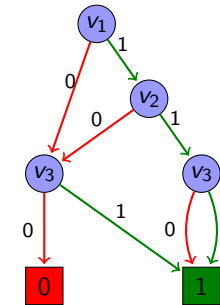
Definition (Shannon reduction)

If both outgoing arcs of an internal node n of a BDD lead to the same node m , then n can be removed from the BDD, with all incoming arcs of n going to m instead.

Reduced ordered BDDs

Reductions

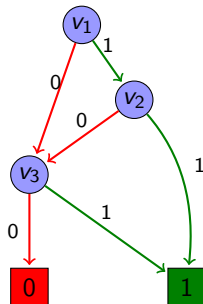
Shannon reduction



Reduced ordered BDDs

Reductions

Shannon reduction



Definition

Definition (reduced ordered BDD)

An ordered BDD is **reduced** iff it does not admit any isomorphism reduction or Shannon reduction.

Theorem (Bryant 1986)

For every state set S and a fixed variable ordering, there exists exactly one reduced ordered BDD representing S .

Moreover, given any ordered BDD B , the equivalent reduced ordered BDD can be computed in linear time in the size of B .

⇝ Reduced ordered BDDs are the canonical representation we were looking for.

From now on, we simply say **BDD** for **reduced ordered BDD**.

Efficient BDD implementation

Ideas

- ▶ Earlier, we showed some BDD performance characteristics.
 - ▶ Example: $S = S'?$ can be tested in time $O(1)$.
- ▶ The critical idea for achieving this performance is to **share structure** not only within a BDD, but also between **different BDDs**.

BDD representation

- ▶ Every BDD (including sub-BDDs) B is represented by a single natural number $id(B)$ called its **ID**.
 - ▶ The zero BDD has ID -2 .
 - ▶ The one BDD has ID -1 .
 - ▶ Other BDDs have IDs ≥ 0 .
- ▶ The BDD operations must satisfy the following invariant: Two BDDs with different ID are **never** identical.

Efficient BDD implementation

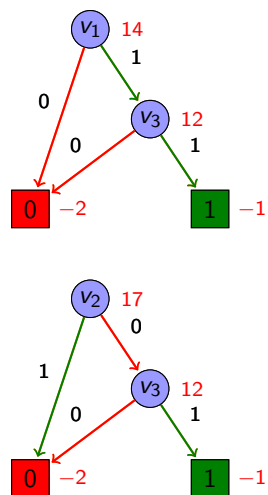
Data structures

Data structures

- ▶ There are three global vectors (dynamic arrays) to represent information on non-sink BDDs with ID $i \geq 0$:
 - ▶ $var[i]$ denotes the decision variable.
 - ▶ $low[i]$ denotes the ID of the 0-successor.
 - ▶ $high[i]$ denotes the ID of the 1-successor.
- ▶ There is some mechanism that keeps track of IDs that are currently unused (garbage collection, reference counting).
 - ▶ This can be implemented without amortized overhead.
- ▶ There is a global hash table *lookup* which maps, for each ID $i \geq 0$ representing a BDD in use, the triple $\langle var[i], low[i], high[i] \rangle$ to i .
 - ▶ Randomized hashing allows constant-time access in the **expected case**. More sophisticated methods allow deterministic constant-time access.

Efficient BDD implementation

Data structures example



formula	ID i	$var[i]$	$low[i]$	$high[i]$
\perp	-2	—	—	—
\top	-1	—	—	—
v_3	12	3	-2	-1
$v_1 \wedge v_3$	14	1	-2	12
$\neg v_2 \wedge v_3$	17	2	12	-2

Core BDD operations

Building the zero BDD

```
def zero():
    return -2
```

Building the one BDD

```
def one():
    return -1
```

Core BDD operations

Building other BDDs

```
def bdd(v: variable, l: ID, h: ID):
  if l = h:
    return l
  if  $\langle v, l, h \rangle \notin \text{lookup}$ :
    Set  $i$  to a new unused ID.
     $\text{var}[i], \text{low}[i], \text{high}[i] := v, l, h$ 
     $\text{lookup}[\langle v, l, h \rangle] := i$ 
  return  $\text{lookup}[\langle v, l, h \rangle]$ 
```

We only create BDDs with **zero**, **one** and **bdd** (i.e., function bdd is the only function writing to *var*, *low*, *high* and *lookup*). Thus:

- ▶ BDDs are guaranteed to be reduced.
- ▶ BDDs with different IDs always represent different sets.

BDD operations

Notations

For convenience, we introduce some additional notations:

- ▶ We define **0** := **zero()**, **1** := **one()**.
- ▶ We write *var*, *low*, *high* as attributes:
 - ▶ **B.var** for $\text{var}[B]$
 - ▶ **B.low** for $\text{low}[B]$
 - ▶ **B.high** for $\text{high}[B]$

Essential vs. derived BDD operations

We distinguish between

- ▶ **essential BDD operations**, which are implemented directly on top of **zero**, **one** and **bdd**, and
- ▶ **derived BDD operations**, which are implemented in terms of the essential operations.

Essential BDD operations

We study the following essential operations:

- ▶ **bdd-includes**(B, s): Test $s \in r(B)$.
- ▶ **bdd-equals**(B, B'): Test $r(B) = r(B')$.
- ▶ **bdd-atom**(a): Build BDD representing $\{s \mid s(a) = 1\}$.
- ▶ **bdd-state**(s): Build BDD representing $\{s\}$.
- ▶ **bdd-union**(B, B'): Build BDD representing $r(B) \cup r(B')$.
- ▶ **bdd-complement**(B): Build BDD representing $\overline{r(B)}$.
- ▶ **bdd-countmodels**(B): Compute $|r(B)|$.
- ▶ **bdd-forget**(B, a): Described later.

Essential operations

Memoization

- ▶ The essential functions are all defined recursively and are free of side effects.
- ▶ We assume (without explicit mention in the pseudo-code) that they all use **dynamic programming** (memoization):
 - ▶ Every **return** statement stores the arguments and result in a memo hash table.
 - ▶ Whenever a function is invoked, the memo is checked if the same call was made previously. If so, the result from the memo is taken to avoid recomputations.
- ▶ The memo may be cleared when the “outermost” recursive call terminates.
 - ▶ The bdd-forget function calls the bdd-union function internally. In this case, the memo for bdd-union may only be cleared once bdd-forget finishes, **not** after each bdd-union invocation finishes.

Memoization is critical for the mentioned runtime bounds.

Essential BDD operations

bdd-includes

Test $s \in r(B)$

```
def bdd-includes(B, s):
    if B = 0:
        return false
    else if B = 1:
        return true
    else if s[B.var] = 1:
        return bdd-includes(B.high, s)
    else:
        return bdd-includes(B.low, s)
```

- ▶ Runtime: $O(k)$
- ▶ This works for partial or full valuations s , as long as all variables appearing in the BDD are defined.

Essential BDD operations

bdd-equals

Test $r(B) = r(B')$

```
def bdd-equals(B, B'):
    return B = B'
```

- ▶ Runtime: $O(1)$

Essential BDD operations

bdd-atom

Build BDD representing $\{s \mid s(a) = 1\}$

```
def bdd-atom(a):
    return bdd(a, 0, 1)
```

- ▶ Runtime: $O(1)$

Essential BDD operations

bdd-state

Build BDD representing $\{s\}$

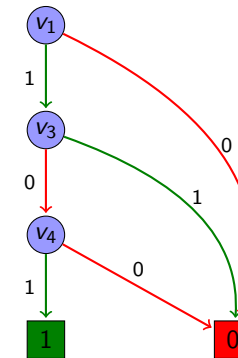
```
def bdd-state(s):
    B := 1
    for each variable v of s, in reverse variable order:
        if s(v) = 1:
            B := bdd(v, 0, B)
        else:
            B := bdd(v, B, 0)
    return B
```

- ▶ Runtime: $O(k)$
- ▶ Works for partial or full valuations s .

Essential BDD operations

bdd-state: Example

$bdd\text{-}state(\{v_1 \mapsto 1, v_3 \mapsto 0, v_4 \mapsto 1\})$



Essential BDD operations

bdd-union

Build BDD representing $r(B) \cup r(B')$

```
def bdd-union(B, B'):
    if B = 0 and B' = 0:
        return 0
    else if B = 1 or B' = 1:
        return 1
    else if B.var < B'.var:
        return bdd(B.var, bdd-union(B.low, B'),
                    bdd-union(B.high, B'))
    else if B.var = B'.var:
        return bdd(B.var, bdd-union(B.low, B'.low),
                    bdd-union(B.high, B'.high))
    else if B.var > B'.var:
        return bdd(B'.var, bdd-union(B, B'.low),
                    bdd-union(B, B'.high))
```

- ▶ Runtime: $O(\|B\| \cdot \|B'\|)$

Essential BDD operations

bdd-complement

Build BDD representing $\overline{r(B)}$

```
def bdd-complement(B):
    if B = 0:
        return 1
    else if B = 1:
        return 0
    else:
        return bdd(B.var, bdd-complement(B.low),
                    bdd-complement(B.high))
```

- ▶ Runtime: $O(\|B\|)$

Essential BDD operations

bdd-countmodels

Compute $|r(B)|$

```

def bdd-countmodels(B):
    return count(B, 0)

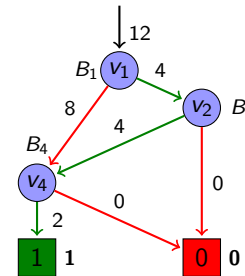
def count(B, i):
    if B = 0:
        return 0
    else if B = 1:
        return  $2^{k-i}$ 
    else:
        Set  $j$  so that  $B.var = v_j$ .
        return  $2^{j-i-1} \cdot (\text{count}(B.\text{low}, j) + \text{count}(B.\text{high}, j))$ 

```

► Runtime: $O(\|B\|)$

Essential BDD operations

bdd-countmodels: Example



BDD represents $v_4 \wedge (\neg v_1 \vee v_2)$ over variables $\{v_1, v_2, v_3, v_4, v_5\}$, i.e. $k = 5$.

```

count(B1, 0) = 1 · (count(B4, 1) + count(B2, 1)) = 12
count(B4, 1) = 4 · (count(0, 4) + count(1, 4)) = 8
count(0, 4) = 0
count(1, 4) = 2
count(B2, 1) = 1 · (count(0, 2) + count(B4, 2)) = 4
count(0, 2) = 0
count(B4, 2) = 2 · (count(0, 4) + count(1, 4)) = 4

```

Essential BDD operations

bdd-forget

The last essential BDD operation is a bit more unusual, but we will need it for defining the semantics of operator application.

Definition (Existential abstraction)

Let A be a set of propositional variables, let S be a set of valuations over A , and let $v \in A$.

The **existential abstraction of v in S** , in symbols $\exists v.S$, is the set of valuations

$$\{s' : (A \setminus \{v\}) \rightarrow \{0, 1\} \mid \exists s \in S : s' \subset s\}$$

over $A \setminus \{v\}$.

Existential abstraction is also called **forgetting**.

Essential BDD operations

bdd-forget

Build BDD representing $\exists v.r(B)$

```

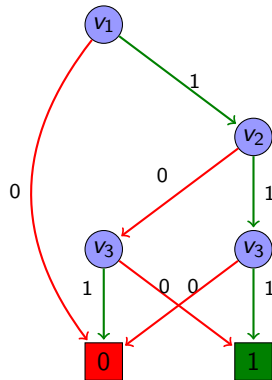
def bdd-forget(B, v):
    if B = 0 or B = 1 or B.var > v:
        return B
    else if B.var < v:
        return bdd(B.var, bdd-forget(B.low, v),
                    bdd-forget(B.high, v))
    else:
        return bdd-union(B.low, B.high)

```

► Runtime: $O(\|B\|^2)$

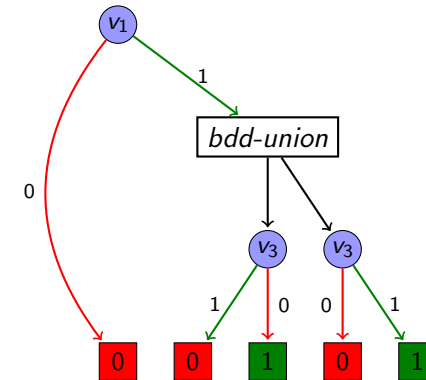
Essential BDD operations

bdd-forget: Example

Forgetting v_2 

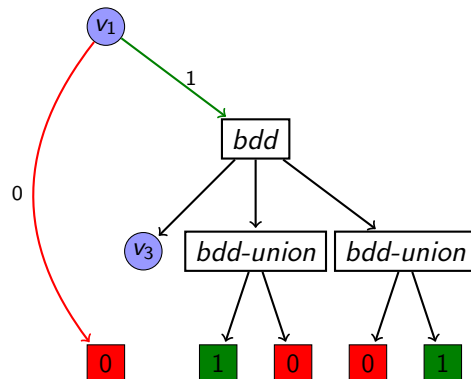
Essential BDD operations

bdd-forget: Example

Forgetting v_2 

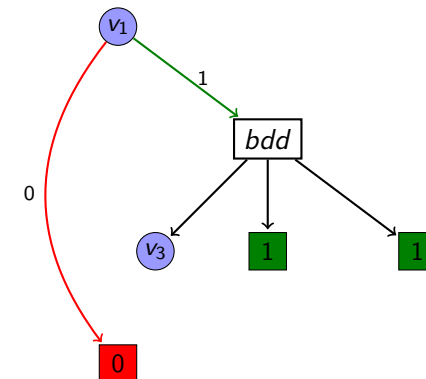
Essential BDD operations

bdd-forget: Example

Forgetting v_2 

Essential BDD operations

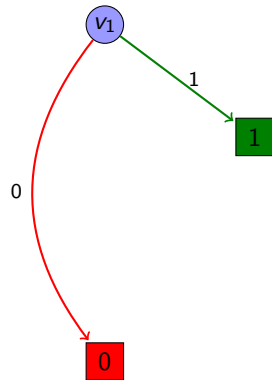
bdd-forget: Example

Forgetting v_2 

Essential BDD operations

bdd-forget: Example

Forgetting v_2



Derived BDD operations

We study the following derived operations:

- ▶ **bdd-intersection**(B, B'):
Build BDD representing $r(B) \cap r(B')$.
- ▶ **bdd-setdifference**(B, B'):
Build BDD representing $r(B) \setminus r(B')$.
- ▶ **bdd-isempty**(B):
Test $r(B) = \emptyset$.
- ▶ **bdd-rename**(B, v, v'):
Build BDD representing $\{ \text{rename}(s, v, v') \mid s \in r(B) \}$, where $\text{rename}(s, v, v')$ is the valuation s with variable v renamed to v' .
 - ▶ If variable v' occurs in B already, the result is undefined.

Derived BDD operations

bdd-intersection, bdd-setdifference

Build BDD representing $r(B) \cap r(B')$

```
def bdd-intersection( $B, B'$ ):
    not-B := bdd-complement( $B$ )
    not-B' := bdd-complement( $B'$ )
    return bdd-complement(bdd-union(not-B, not-B'))
```

Build BDD representing $r(B) \setminus r(B')$

```
def bdd-setdifference( $B, B'$ ):
    return bdd-intersection( $B, \text{bdd-complement}(B')$ )
```

- ▶ Runtime: $O(\|B\| \cdot \|B'\|)$
- ▶ These functions can also be easily implemented directly, following the structure of *bdd-union*.

Derived BDD operations

bdd-isempty

Test $r(B) = \emptyset$

```
def bdd-isempty( $B$ ):
    return bdd-equals( $B, 0$ )
```

- ▶ Runtime: $O(1)$

Derived BDD operations

bdd-rename

Build BDD representing $\{ \text{rename}(s, v, v') \mid s \in r(B) \}$

```
def bdd-rename( $B, v, v'$ ):
     $v\text{-and-}v' := \text{bdd-intersection}(\text{bdd-atom}(v), \text{bdd-atom}(v'))$ 
     $\text{not-}v := \text{bdd-complement}(\text{bdd-atom}(v))$ 
     $\text{not-}v' := \text{bdd-complement}(\text{bdd-atom}(v'))$ 
     $\text{not-}v\text{-and-not-}v' := \text{bdd-intersection}(\text{not-}v, \text{not-}v')$ 
     $v\text{-eq-}v' := \text{bdd-union}(v\text{-and-}v', \text{not-}v\text{-and-not-}v')$ 
    return  $\text{bdd-forget}(\text{bdd-intersection}(B, v\text{-eq-}v'), v)$ 
```

- Runtime: $O(\|B\|^2)$

Derived BDD operations

bdd-rename: Remarks

- Renaming sounds like a simple operation.
- Why is it so expensive?

This is **not** because the algorithm is bad:

- Renaming **must** take at least quadratic time:
 - There exist families of BDDs B_n with k variables such that renaming v_1 to v_{k+1} increases the size of the BDD from $\Theta(n)$ to $\Theta(n^2)$.
- However, renaming is cheap in **some cases**:
 - For example, renaming to a **neighboring** unused variable (e.g. from v_i to v_{i+1}) is always possible in linear time by simply relabeling the decision variables of the BDD.
- In practice, one can usually choose a variable ordering where renaming only occurs between neighboring variables.

Breadth-first search with progression and BDDs

Progression breadth-first search

```
def bfs-progression( $A, I, O, G$ ):
     $\text{goal} := \text{formula-to-set}(G)$ 
     $\text{reached} := \{I\}$ 
    loop:
        if  $\text{reached} \cap \text{goal} \neq \emptyset$ :
            return solution found
         $\text{new-reached} := \text{reached} \cup \text{apply}(\text{reached}, O)$ 
        if  $\text{new-reached} = \text{reached}$ :
            return no solution exists
         $\text{reached} := \text{new-reached}$ 
```

Breadth-first search with progression and BDDs

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def bfs-progression( $A, I, O, G$ ):
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         $\text{reached} := \text{new-reached}$ 
```

Use *bdd-atom*, *bdd-complement*, *bdd-union*, *bdd-intersection*.

Breadth-first search with progression and BDDs

Progression breadth-first search

```

def bfs-progression(A, I, O, G):
    goal := formula-to-set(G)
    reached := {I}
    loop:
        if reached ∩ goal ≠ ∅:
            return solution found
        new-reached := reached ∪ apply(reached, O)
        if new-reached = reached:
            return no solution exists
        reached := new-reached

```

Use *bdd-state*.

Breadth-first search with progression and BDDs

Progression breadth-first search

```

def bfs-progression(A, I, O, G):
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    reached := {I}
    loop:
        if reached ∩ goal ≠ ∅:
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        new-reached := reached ∪ apply(reached, O)
        if new-reached = reached:
            return no solution exists
        reached := new-reached

```

Use *bdd-intersection*, *bdd-isempty*.

Breadth-first search with progression and BDDs

Progression breadth-first search

```

def bfs-progression(A, I, O, G):
    goal := formula-to-set(G)
    reached := {I}
    loop:
        if reached ∩ goal ≠ ∅:
            return solution found
        new-reached := reached ∪ apply(reached, O)
        if new-reached = reached:
            return no solution exists
        reached := new-reached

```

Use *bdd-union*.

Breadth-first search with progression and BDDs

Progression breadth-first search

```

def bfs-progression(A, I, O, G):
    goal := formula-to-set(G)
    reached := {I}
    loop:
        if reached ∩ goal ≠ ∅:
            return solution found
        new-reached := reached ∪ apply(reached, O)
        if new-reached = reached:
            return no solution exists
        reached := new-reached

```

Use *bdd-equals*.

Breadth-first search with progression and BDDs

Progression breadth-first search

```

def bfs-progression( $A, I, O, G$ ):
     $goal := formula-to-set(G)$ 
     $reached := \{I\}$ 
    loop:
        if  $reached \cap goal \neq \emptyset$ :
            return solution found
         $new-reached := reached \cup apply(reached, O)$ 
        if  $new-reached = reached$ :
            return no solution exists
         $reached := new-reached$ 

```

How to do this?

The *apply* function

- ▶ We need an operation that, for a set of states *reached* (given as a BDD) and a set of operators *O*, computes the set of states (as a BDD) that can be reached by applying some operator $o \in O$ in some state $s \in reached$.
- ▶ We have seen something similar already. . .

Translating operators into formulae

(slide taken from the “planning by satisfiability testing” chapter)

Definition (operators in propositional logic)

Let $o = \langle c, e \rangle$ be an operator and A a set of state variables.

Define $\tau_A(o)$ as the conjunction of

$$c \quad (1)$$

$$\bigwedge_{a \in A} (EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))) \leftrightarrow a' \quad (2)$$

$$\bigwedge_{a \in A} \neg (EPC_a(e) \wedge EPC_{\neg a}(e)) \quad (3)$$

Condition (1) states that the precondition of o is satisfied.

Condition (2) states that the **new value of a** , represented by a' , is 1 if the old value was 1 and it did not become 0, or if it became 1.

Condition (3) states that none of the state variables is assigned both 0 and 1. Together with (1), this encodes applicability of the operator.

The *apply* function

- ▶ The formula $\tau_A(o)$ describes the applicability of a **single** operator o and the effect of applying o as a binary formula over variables A (describing the state in which o is applied) and A' (describing the resulting state).
- ▶ The formula $\bigvee_{o \in O} \tau_A(o)$ describes state transitions by **any** operator.
- ▶ We can translate this formula to a BDD (over variables $A \cup A'$) using **bdd-atom**, **bdd-complement**, **bdd-union**, **bdd-intersection**.
- ▶ The resulting BDD is called the **transition relation** of the planning task, written as $T_A(O)$.

The *apply* function

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

The *apply* function

```
def apply(reached, O):
    B := TA(O)
    B := bdd-intersection(B, reached)
    for each a ∈ A:
        B := bdd-forget(B, a)
    for each a ∈ A:
        B := bdd-rename(B, a', a)
    return B
```

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    return B
```

This describes the set of **state pairs** $\langle s, s' \rangle$ where s' is a successor of s in terms of variables $A \cup A'$.

The *apply* function

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

The *apply* function

```
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    for each a ∈ A:
        B := bdd-rename(B, a', a)
    return B
```

This describes the set of state pairs $\langle s, s' \rangle$ where s' is a successor of s **and** $s \in \text{reached}$ in terms of variables $A \cup A'$.

The *apply* function

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

The *apply* function

```
def apply(reached, O):
    B := TA(O)
    B := bdd-intersection(B, reached)
    for each a ∈ A:
        B := bdd-forget(B, a)
    for each a ∈ A:
        B := bdd-rename(B, a', a)
    return B
```

This describes the set of states s' which are successors **of some state** $s \in \text{reached}$ in terms of variables A' .

The *apply* function

Using the transition relation, we can compute *apply(reached, O)* as follows:

The *apply* function

```
def apply(reached, O):
    B := TA(O)
    B := bdd-intersection(B, reached)
    for each a ∈ A:
        B := bdd-forget(B, a)
    for each a ∈ A:
        B := bdd-rename(B, a', a)
    return B
```

This describes the set of states s' which are successors of some state $s \in \text{reached}$ in terms of variables A .

The *apply* function

Using the transition relation, we can compute *apply(reached, O)* as follows:

The *apply* function

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    for each a ∈ A:
        B := bdd-rename(B, a', a)
    return B
```

Thus, *apply* indeed computes the set of successors of *reached* using operators O .

Planning with BDDs

Summary and conclusion

- ▶ Binary decision diagrams are a data structure to compactly represent and manipulate sets of valuations.
- ▶ They can be used to implement a blind breadth-first search algorithm in an efficient way.

Planning with BDDs

Performance

- ▶ For good performance, we need a good variable ordering.
 - ▶ Variables that refer to the same state variable before and after operator application (a and a') should be neighbors in the transition relation BDD.
- ▶ Use mutexes to reformulate as a multi-valued task.
 - ▶ Use $\lceil \log_2 n \rceil$ BDD variables to represent a variable with n possible values.

With these two ideas, performance is not bad for an algorithm that generates optimal (sequential) plans.

Planning with BDDs

Outlook

Is this all there is to it?

- ▶ For classical deterministic planning, **almost**.
 - ▶ Practical implementations also perform **regression** or **bidirectional** searches.
 - ▶ This is only a minor modification.
- ▶ However, BDDs are more commonly used for **non-deterministic** planning.
- ▶ More about this later.