An Introduction to Game Theory
Part V:
Extensive Games with Perfect Information
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Motivation

• So far, all games consisted of just one simultaneous move by all players
• Often, there is a whole sequence of moves and player can react to the moves of the other players
• Examples:
  – board games
  – card games
  – negotiations
  – interaction in a market
Example: Entry Game

- An *incumbent* faces the possibility of *entry* by a *challenger*. The *challenger* may enter (*in*) or not enter (*out*). If it enters, the *incumbent* may either *give in* or *fight*.

- The payoffs are
  - challenger: 1, incumbent: 2 if challenger does not enter
  - challenger: 2, incumbent: 1 if challenger enters and incumbent gives in
  - challenger: 0, incumbent: 0 if challenger enters and incumbent fights

(similar to chicken – but here we have a sequence of moves!)
Formalization: Histories

• The possible developments of a game can be described by a *game tree* or a mechanism to construct a game tree

• Equivalently, we can use the set of paths starting at the root: all potential *histories* of moves
  – potentially infinitely many (infinite branching)
  – potentially infinitely long
Extensive Games with Perfect Information

An **extensive games with perfect information** consists of

- a non-empty, finite set of players $N = \{1, \ldots, n\}$
- a set $H$ (histories) of sequences such that
  - $\langle \rangle \in H$
  - $H$ is prefix-closed
  - if for an infinite sequence $\langle a_i \rangle_{i \in N}$ every prefix of this sequence is in $H$, then the infinite sequence is also in $H$
  - sequences that are not a proper prefix of another strategy are called **terminal histories** and are denoted by $Z$. The elements in the sequences are called **actions**.
- a player function $P: H \setminus Z \rightarrow N$,
- for each player $i$ a payoff function $u_i: Z \rightarrow \mathbb{R}$

- A game is **finite** if $H$ is finite
- A game as a **finite horizon**, if there exists a finite upper bound for the length of histories
Entry Game – Formally

- players $N = \{1, 2\}$ (1: challenger, 2: incumbent)
- histories $H = \{\langle \rangle, \langle \text{out} \rangle, \langle \text{in} \rangle, \langle \text{in, fight} \rangle, \langle \text{in, give_in} \rangle\}$
- terminal histories: $Z = \{\langle \text{out} \rangle, \langle \text{in, fight} \rangle, \langle \text{in, give_in} \rangle\}$
- player function:
  - $P(\langle \rangle) = 1$
  - $P(\langle \text{in} \rangle) = 2$
- payoff function
  - $u_1(\langle \text{out} \rangle) = 1$, $u_2(\langle \text{out} \rangle) = 2$
  - $u_1(\langle \text{in, fight} \rangle) = 0$, $u_2(\langle \text{in, fight} \rangle) = 0$
  - $u_1(\langle \text{in, give_in} \rangle) = 2$, $u_2(\langle \text{in, give_in} \rangle) = 1$
Strategies

- The number of possible actions after history h is denoted by $A(h)$.
- A strategy for player $i$ is a function $s_i$ that maps each history $h$ with $P(h) = i$ to an element of $A(h)$.
- *Notation:* Write strategy as a sequence of actions as they are to be chosen at each point when visiting the nodes in the game tree in breadth-first manner.

- Possible strategies for player 1:
  - AE, AF, BE, BF
- for player 2:
  - C, D
- Note: Also decisions for histories that cannot happen given earlier decisions!
Outcomes

- The outcome $O(s)$ of a strategy profile $s$ is the terminal history that results from applying the strategies successively to the histories starting with the empty one.
- What is the outcome for the following strategy profiles?
  - $O(AF,C) =$
  - $O(AF,D) =$
  - $O(BF,C) =$
Nash Equilibria in Extensive Games with Perfect Information

- A strategy profile $s^*$ is a Nash Equilibrium in an extensive game with perfect information if for all players $i$ and all strategies $s_i$ of player $i$:
  \[ u_i(O(s_{-i}^*, s_i^*)) \geq u_i(O(s_{-i}^*, s_i)) \]

- Equivalently, we could define the strategic form of an extensive game and then use the existing notion of Nash equilibrium for strategic games.
The Entry Game - again

- Nash equilibra?
  - In, Give in
  - Out, Fight

- But why should the challenger take the “threat” seriously that the incumbent starts a fight?

- Once the challenger has played “in”, there is no point for the incumbent to reply with “fight”. So “fight” can be regarded as an empty threat

<table>
<thead>
<tr>
<th></th>
<th>Give in</th>
<th>Fight</th>
</tr>
</thead>
<tbody>
<tr>
<td>In</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Out</td>
<td>1,2</td>
<td>1,2</td>
</tr>
</tbody>
</table>

- Apparently, the Nash equilibrium out, fight is not a real “steady state” – we have ignored the sequential nature of the game
Sub-games

- Let $G=(N,H,P,(u_i))$ be an extensive game with perfect information. For any non-terminal history $h$, the sub-game $G(h)$ following history $h$ is the following game: $G'=(N,H',P',(u_i'))$ such that:
  - $H'$ is the set of histories such that for all $h'$: $(h,h') \in H$
  - $P'(h') = P((h,h'))$
  - $u_i'(h') = u_i((h,h'))$

How many sub-games are there?
Applying Strategies to Sub-games

- If we have a strategy profile $s^*$ for the game $G$ and $h$ is a history in $G$, then $s^*|_h$ is the strategy profile after history $h$, i.e., it is a strategy profile for $G(h)$ derived from $s^*$ by considering only the histories following $h$.

- For example, let $((\text{out}), (\text{fight}))$ be a strategy profile for the entry game. Then $((\text{)}, (\text{fight}))$ is the strategy profile for the sub-game after player 1 played “in”.

Sub-game Perfect Equilibria

• A sub-game perfect equilibrium (SPE) of an extensive game with perfect information is a strategy profile $s^*$ such that for all histories $h$, the strategies in $s^*|_h$ are optimal for all players.

• Note: ((out), (fight)) is not a SPE!

• Note: A SPE could also be defined as a strategy profile that induces a NE in every sub-game
Example: Distribution Game

- Two objects of the same kind shall be distributed to two players. Player 1 suggest a distribution, player 2 can accept (+) or reject (-). If she accepts, the objects are distributed as suggested by player 1. Otherwise nobody gets anything.
- NEs?
- SPEs?

- ((2,0),+xx) are NEs
- ((2,0),--x) are NEs
- ((1,1),-+x) are NEs
- ((0,1),--+) is a NE

Only
- ((2,0),+++) is a SPE
- ((1,1),-+++) is a SPE
Existence of SPEs

• **Infinite games** may not have a SPE
  – Consider the 1-player game with actions \([0,1)\) and payoff \(u_1(a) = a\).

• If a game **does not have a finite horizon**, then it may not possess an SPE:
  – Consider the 1-player game with infinite histories such that the infinite histories get a payoff of 0 and all finite prefixes extended by a termination action get a payoff that is proportional to their length.
Finite Games Always Have a SPE

- Length of a sub-game = length of longest history
- Use **backward induction**
  - Find the optimal play for all sub-games of length 1
  - Then find the optimal play for all sub-games of length 2 (by using the above results)
  - ...
  - until length $n = \text{length of game}$
  - game has an SPE
- SPE is not necessarily unique – agent may be indifferent about some outcomes
- All SPEs can be found this way!
Strategies and Plans of Action

- Strategies contain decisions for **unreachable** situations!
- Why should player 1 worry about the choice after A,C if he will play B?
- Can be thought of as
  - player 2’s beliefs about player 1
  - what will happen if by mistake player 1 chooses A
The Distribution Game - again

- Now it is easy to find all SPEs
- Compute optimal actions for player 2
- Based on the results, consider actions of player 1
Another Example:
The Chain Store Game

- If we play the entry game for \( k \) periods and add up the payoff from each period, what will be the SPEs?
- By backward induction, the only SPE is the one, where in every period (in, give_in) is selected.
- However, for the incumbent, it could be better to play sometimes fight in order to “build up a reputation” of being aggressive.
Yet Another Example: The Centipede Game

• The players move alternately
• Each prefers to stop in his move over the other player stopping in the next move
• However, if it is not stopped in these two periods, this is even better
• What is the SPE?
Centipede: Experimental Results

- This game has been played ten times by 58 students facing a new opponent each time.
- With experience, games become shorter.
- However, far off from Nash equilibrium.
Relationship to *Minimax*

- **Similarities to *Minimax***
  - solving the game by searching the game tree bottom-up, choosing the optimal move at each node and propagating values upwards

- **Differences**
  - More than two players are possible in the backward-induction case
  - Not just one number, but an entire payoff profile

- **So, is *Minimax* just a special case?**
Possible Extensions

• One could add **random moves** to extensive games. Then there is a special player which chooses its actions randomly
  – SPEs still exist and can be found by backward induction. However, now the expected utility has to be optimized

• One could add **simultaneous moves**, that the players can sometimes make moves in parallel
  – SPEs might not exist anymore (simple argument!)

• One could add **“imperfect information”**: The players are not always informed about the moves other players have made.
Conclusions

- Extensive games model games in which more than one simultaneous move is allowed.
- The notion of Nash equilibrium has to be refined in order to exclude implausible equilibria – those with empty threats.
- Sub-game perfect equilibria capture this notion.
- In finite games, SPEs always exist.
- All SPEs can be found by using backward induction.
- Backward induction can be seen as a generalization of the Minimax algorithm.
- A number of plausible extensions are possible: simultaneous moves, random moves, imperfect information.