Motivation

- So far, we assumed that all players have perfect knowledge about the preferences (the payoff function) of the other players.
- Often unrealistic.
- For example, in auctions people are not sure about the valuations of the others. – what to do in a sealed bid auction?
Example

• Let’s assume the BoS game, where player 1 is not sure, whether player 2 wants to meet her or to avoid her,
• She assumes a probability of 0.5 for each case.
• Player 2 knows the preferences of player 1
Example (cont.)

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<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
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<tbody>
<tr>
<td>Bach</td>
<td>2,1</td>
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Prob. 0.5

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Prob. 0.5
What is the Payoff?

- Player 1 views player 2 as being one of two possible *types*
- Each of these types may make an independent decision
- So, the friendly player 2 may choose B and the unfriendly one S: (B,S)
- **Expected payoff** when player 1 plays B:
  \[ 0.5 \times 2 + 0.5 \times 0 = 1 \]
A **Nash equilibrium** in pure strategies is a triple \((x,(y,z))\) of actions such that:

- the action \(x\) of player 1 is optimal given the actions \((y,z)\) of both types of player 2 and the belief about the state
- the actions \(y\) and \(z\) of each type of player 2 are optimal given the action \(x\) of player 1

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<thead>
<tr>
<th></th>
<th>(B,B)</th>
<th>(B,S)</th>
<th>(S,B)</th>
<th>(S,S)</th>
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<tr>
<td>B</td>
<td>2 (1,0)</td>
<td>1 (1,2)</td>
<td>1 (0,0)</td>
<td>0 (0,2)</td>
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<td>S</td>
<td>0 (0,1)</td>
<td>0.5 (0,0)</td>
<td>0.5 (2,1)</td>
<td>1 (2,0)</td>
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Nash Equilibria?

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<td>0.5 (0,0)</td>
<td>0.5 (2,1)</td>
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- Is there a Nash equilibrium?
  - Yes: B, (B,S)

- Is there a NE where player 1 plays S?
  - No
Formalization: States and Signals

- There are **states**, which completely determine the preferences / payoff functions
  - In our example: *friendly* and *unfriendly*

- Before the game starts, each player receives a **signal** that tells her something about the state
  - In our example:
    - Player 2 receives a state, which type she is
    - Player 1 gets no information about the state and has only her beliefs about probabilities.

- Although, the actions for non-realized types of player 2 are irrelevant for player 2, they are necessary for player 1 (and therefore also for player 2) when deliberating about possible action profiles and their payoffs.
General Bayesian Games

- A Bayesian game consists of
  - a set of players $N = \{1, \ldots, n\}$
  - a set of states $\Omega = \{\omega_1, \ldots, \omega_k\}$

- and for each player $i$
  - a set of actions $A_i$
  - a set of signals $T_i$ and a signal function $\tau_i: \Omega \to T_i$
  - for each signal a belief about the possible states (a probability distribution over the states associated with the signal) $Pr(\omega \mid t_i)$
  - a payoff function $u_i(a, \omega)$ over pairs of action profiles and states, where the expected value for $a_i$ represents the preferences:
    $\sum_{\omega \in \Omega} Pr(\omega \mid t_i) u_i((a_i, \hat{a}_i(\omega)), \omega)$
    with $\hat{a}_i(\omega)$ denoting the choice by $i$ when she has received the signal $\tau_i(\omega)$
Example: BoS with Uncertainty

- **Players**: \{1, 2\}
- **States**: \{friendly, unfriendly\}
- **Actions**: \{B, S\}
- **Signals**: \(T=\{a,b,c\}\)
  - \(\tau_1(\omega_i) = a\) for \(i=1,2\)
  - \(\tau_2(\text{friendly}) = b, \tau_2(\text{unfriendly}) = c\),
- **Beliefs**:
  - \(Pr(\text{friendly} | a) = 0.5, Pr(\text{unfriendly} | a) = 0.5\)
  - \(Pr(\text{friendly} | b) = 1, Pr(\text{friendly} | b) = 0\)
  - \(Pr(\text{friendly} | c) = 0, Pr(\text{friendly} | c) = 1\)
- **Payoffs**: As in the left and right tables *on the slide*
Example: Information can hurt

- In single-person games, knowledge can never hurt, but here it can!
- Two players, both don’t know which state und consider both states $\omega_1$ and $\omega_2$ as equally probable (0.5)
- Note: Preferences of player 1 are known, while the preferences of player 2 are unknown (to both!)

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>L</th>
<th>M</th>
<th>R</th>
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<tbody>
<tr>
<td>T</td>
<td>3,2</td>
<td>3,0</td>
<td>3,3</td>
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<tr>
<td>B</td>
<td>6,6</td>
<td>0,0</td>
<td>0,9</td>
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<tr>
<th>$\omega_2$</th>
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Example (cont.)

- Player 2’s unique best response is: L
- For this reason, player 1 will play B
- Payoff: 6,6 – only NE, even when mixed strategies!
- When player 2 can distinguish the states, R and M are dominating actions
- \((T,(R,M))\) is the unique NE

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Incentives and Uncertain Knowledge May Lead to Suboptimal Solutions

- $\tau_1(\alpha) = a, \tau_1(\beta) = b, \tau_1(\gamma) = b$
  - $Pr(\alpha | a) = 1$
  - $Pr(\beta | b) = 0.75, Pr(\gamma | b) = 0.25$
- $\tau_2(\alpha) = c, \tau_2(\beta) = c, \tau_2(\gamma) = d$
  - $Pr(\alpha | c) = 0.75, Pr(\beta | c) = 0.25$
  - $Pr(\gamma | d) = 1$

- In state $\gamma$, there are 2 NEs
- In state $\gamma$, player 2 knows her preferences, but player 1 does not know that!
- The incentive for player 1 to play R in state $\alpha$ "infects" the game and only (R,R),(R,R) is an NE

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<tr>
<td>$\beta &amp; \gamma$</td>
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The Infection

- Player 1 must play R when receiving signal a (= state $\alpha$)!
- Player 2 will therefore never play L when receiving c (= $\alpha$ or $\beta$)
- For this reason, player 1 will never play L when receiving b (= $\beta$ or $\gamma$)
- Therefore player 2 will also play R when receiving d (= $\gamma$)
- Therefore the unique NE is ((R,R),(R,R))!

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$\tau_1(a) = a$, $\tau_1(b) = b$, $\tau_1(\gamma) = b$
$Pr(a|a) = 1$
$Pr(\beta|b) = 0.75$, $Pr(\gamma|b) = 0.25$

$\tau_2(\alpha) = c$, $\tau_2(\beta) = c$, $\tau_2(\gamma) = d$
$Pr(a|c) = 0.75$, $Pr(\beta|c) = 0.25$
$Pr(\gamma|d) = 1$
Auctions with Imperfect Information

- Players: \( N = \{1, \ldots, n\} \)
- States: the set of all profiles of valuations \((v_1, \ldots, v_n)\), where \(0 \leq v_i \leq v_{\text{max}}\)
- Actions: Set of possible bids
- Signals: The set of the player \(i\)'s valuation \(\tau_i(v_1, \ldots, v_n) = v_i\)
- Beliefs: \(F(v)\) is the probability that the other bidder values of the object is at most \(v\), i.e., \(F(v_1) \times \cdots \times F(v_{i-1}) \times F(v_{i+1}) \times \cdots \times F(v_n)\) is the probability, that all other players \(j \neq i\) value the object at most \(v_j\)
- Payoff: \(u_i(b,(v_1, \ldots, v_n)) = (v_i - P(b))/m\) if \(b_j \leq b\) for all \(i \neq j\) and \(b_j = b\) for \(m\) players and \(P(b)\) being the price function:
  - \(P(b)\) the highest bid = first price auction
  - \(P(b)\) the second highest bid = second price auction
Private and Common Values

- If the valuations are *private*, that is each one cares only about the his one appreciation (e.g., in art),
  - valuations are completely independent
  - one does not gain information when people submit public bids
- In an auction with *common valuations*, which means that the players share the value system but may be unsure about the real value (antiques, technical devices, exploration rights),
  - valuations are not independent
  - one might gain information from other players bids
- Here we consider private values
Second Price Sealed Bid Auction

- \( P(b) \) is what the second highest bid was
- As in the perfect information case
  - It is a weakly dominating action to bid one's own valuation \( v_i \)
  - There exist other, non-efficient, equilibria
First Price Sealed Bid Auction

- A bid of $v_i$ weakly dominates any bid higher than $v_i$
- A bid of $v_i$ does not weakly dominate a bid lower than $v_i$
- A bid lower than $v_i$ weakly dominates $v_i$
- NE probably at a point below $v_i$
- General analysis is quite involved
- Simplifications:
  - only 2 players
  - $v_{\text{max}} = 1$
  - uniform distribution of valuations, i.e., $F(v) = v$
First Price Sealed Bid Auction (2)

- Let $B_i(v)$ the bid of type $v$ for player $i$.
- **Claim**: Under the mentioned conditions, the game has a NE for $B_i(v) = v/2$.
- Assume that player 2 bids this way, then as far as player 1 is concerned, player 2’s bids are uniformly distributed between 0 and 0.5.
- Thus, if player 1 bids $b_1 > 0.5$, she wins. Otherwise, the probability that she wins is $F(2b_1)$.
- The payoff is
  - $v_1 - b_1$ if $b_1 > 0.5$
  - $2b_1 (v_1 - b_1) = 2b_1 v_1 - 2b_1^2$ if $0 \leq b_1 \leq 0.5$
In other words, \( 0.5v_1 \) is the best response to \( B_2(v) = v/2 \) for player 1.

Since the players are symmetric, this also holds for player 2.

Hence, this is a NE.

In general, for \( m \) players, the NE is \( B_i(v) = v/m \) for \( m \) players.

Can also be shown for general distributions.
Conclusion

- If the players are not fully informed about their own and others' utilities, we have imperfect information.
- The technical tool to model this situation are Bayesian games.
- New concepts are states, signals, beliefs and expected utilities over the believed distributions over states.
- Being informed can hurt!
- Auctions are more complicated in the imperfect information case, but can still be solved.