An Introduction to Game Theory
Part II: Mixed and Correlated Strategies
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Randomizing Actions …

- Since there does not seem to exist a rational decision, it might be best to randomize strategies.
- Play Head with probability $p$ and Tail with probability $1-p$
- Switch to expected utilities

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Some Notation

- Let $G = (N, (A_i), (u_i))$ be a strategic game
- Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$ – the set of mixed strategies $\alpha_i \in \Delta(A_i)$
- $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$
- A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \prod_i \alpha_i(a_i)$
- The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) \cdot u_i(a)$
Example of a Mixed Strategy

- Let
  - $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$
  - $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$

- Then
  - $p(H,H) = 2/9$
  - ...
  - $U_1(\alpha_1, \alpha_2) = ?$
  - $U_2(\alpha_1, \alpha_2) = ?$

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Mixed Extensions

- The mixed extension of the strategic game \((N, (A_i), (u_i))\) is the strategic game \((N, \Delta(A_i), (U_i))\).

- The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.

- Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.
Nash’s Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

– Note that it is essential that the game is finite
– So, there exists always a solution
– What is the computational complexity?
– This is an open problem! Not known to be NP-hard, but there is no known polynomial time algorithm
– Identifying a NE with a value larger than a particular value is NP-hard
The Support

• We call all pure actions $a_i$ that are chosen with non-zero probability by $\alpha_i$ the support of the mixed strategy $\alpha_i$.

**Lemma.** Given a finite strategic game, $\alpha^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_{-i}^*$. 
Proving the Support Lemma

→ Assume that $\alpha^*$ is a Nash equilibrium with $a_i$ being in the support of $\alpha_i^*$ but not being a best response to $\alpha_{-i}^*$.  
  • This means, by reassigning the probability of $a_i$ to the other actions in the support, one can get a higher payoff for player $i$.  
  • This implies $\alpha^*$ is not a Nash equilibrium $\square$ contradiction  
← (Proving the contraposition): Assume that $\alpha^*$ is not a Nash equilibrium.  
  • This means that there exists $a_i'$ that is a better response than $\alpha_i^*$ to $\alpha_{-i}^*$.  
  • Then because of how $U_i$ is computed, there must be an action $a_i'$ in the support of $\alpha_i'$ that is a better response (higher utility) to $\alpha_{-i}^*$ than a pure action $a_i^*$ in the support of $\alpha_i^*$.  
  • This implies that there are actions in the support of $\alpha_i^*$ that are not best responses to $\alpha_{-i}^*$. 
Using the Support Lemma

- The **Support Lemma** can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
  - There are **4 potential Nash equilibria** in pure strategies
    - *Easy to check*
  - There are another **4 potential Nash equilibrium types** with a 1-support (pure) against 2-support mixed strategies
    - Exists only if the corresponding pure strategy profiles are already Nash equilibria (follows from **Support Lemma**)
  - There exists one other potential Nash equilibrium type with a 2-support against a 2-support mixed strategies
    - Here we can use the **Support Lemma** to compute an NE (if there exists one)
1-Support Against 2-Support

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<th>L</th>
<th>R</th>
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<tr>
<td>T</td>
<td>5,5</td>
<td>5,5</td>
</tr>
<tr>
<td>B</td>
<td>-100,6</td>
<td>6,1</td>
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- There is one NE in pure strategies: (T,L)
- There are many mixed NEs of type $\alpha_1(T) = 1$ and $\alpha_2(L), \alpha_2(R) > 0$
- It is clear that one of L or R must form a NE together with T!

- Assume mixed NE with first strategy (T) of player one as pure strategy:
  - $U_1((1,0), (\alpha_2(L), \alpha_2(R))) \geq U_1((0,1), (\alpha_2(L), \alpha_2(R)))$
  - $u_1(T,L)\alpha_2(L) + u_1(T,R)\alpha_2(R) \geq u_1(B,L)\alpha_2(L) + u_1(B,R)\alpha_2(R)$
- Because of this inequation, it follows that either:
  - $u_1(T,L) \geq u_1(B,L)$ or
  - $u_1(T,R) \geq u_1(B,R)$
- Since it is NE, it is clear that
  - $u_2(T,L) = u_2(T,R)$
- Hence, either T,L or T,R must be a NE
A Mixed Nash Equilibrium for Matching Pennies

There is clearly no NE in pure strategies. Let's try whether there is a NE \( \alpha^* \) in mixed strategies.

Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy \( \alpha^{-1} \).

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\[
U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))
\]

\[
U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)
\]

\[
U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)
\]

\[
\alpha_2(H) - \alpha_2(T) = -\alpha_2(H) + \alpha_2(T)
\]

\[
2\alpha_2(H) = 2\alpha_2(T)
\]

\[
\alpha_2(H) = \alpha_2(T)
\]

Because of \( \alpha_2(H) + \alpha_2(T) = 1 \):

- \( \alpha_2(H) = \alpha_2(T) = 1/2 \)
- Similarly for player 1!

\[
U_1(\alpha^*) = 0
\]
### Mixed NE for BoS

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<th>Bach</th>
<th>Stravinsky</th>
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<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
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- There are obviously 2 NEs in pure strategies.
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

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<th>(U_1((1,0), (\alpha_2(B), \alpha_2(S)))) =</th>
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<td>(2\alpha_2(B) + 0\alpha_2(S))</td>
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Because of \(\alpha_2(B) + \alpha_2(S) = 1\):
- \(\alpha_2(B) = 1/3\)
- \(\alpha_2(S) = 2/3\)

- Similarly for player 1!

- \(U_1(\alpha^*) = 2/3\)
Couldn’t we Help the BoS Players?

- BoS have two pure strategy Nash equilibria – but which should they play?
- They can play a mixed strategy, but this is worse than any pure strategy
- The solution is to talk about, where to go
- Use an external random signal to decide where to go
  - Correlated Nash equilibria
  - In the BoS case, we get a payoff of 1.5
The 2/3 of Average Game

• You have \( n \) players that are allowed to choose a number between 1 and \( K \).
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would you play?
• What mixed strategy would you play?

Are there NEs in pure and/or mixed strategies?

Let’s play it: Please write down a number between 1 and 100.
A Nash Equilibrium in Pure Strategies

- All playing 1 is a NE in pure strategies
  - A deviation does not make sense
- All playing the same number different from 1 is not a NE
  - Choosing the number just below gives you more
- Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.

So: Why did you not choose 1?

Perhaps you acted rationally by assuming that the others do not act rationally?
Are there Proper Mixed Strategy Nash Equilibria?

- Assume there exists a mixed NE \( \alpha \) different from the pure NE (1,1,...,1)
- Then there exists a maximal \( k^* > 1 \) which is played by some player with a probability > 0.
  - Assume player \( i \) does so, i.e., \( k^* \) is in the support of \( \alpha_i \).
- This implies \( U_i(k^*,\alpha_{-i}) > 0 \), since \( k^* \) should be as good as all the other strategies of the support.
- Let \( a \) be a realization of \( \alpha \) s.t. \( u_i(a) > 0 \). Then at least one other player must play \( k^* \), because not all others could play below 2/3 of the average!
- In this situation player \( i \) could get more by playing \( k^*-1 \).
- This means, playing \( k^*-1 \) is better than playing \( k^* \), i.e., \( k^* \) cannot be in the support, i.e., \( \alpha \) cannot be a NE
Conclusion

• Although Nash equilibria do not always exist, one can give a guarantee, when we randomize finite games:
  ➢ For every finite strategic game, there exists a Nash equilibrium in mixed strategies
• Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff — Support Lemma
• The Support Lemma can be used to determine mixed strategy NEs for 2-person games with 2x2 action sets
• In general, there is no poly-time algorithm known for finding one Nash equilibrium (and identifying one with a given strictly positive payoff is NP-hard)
• In addition to pure and mixed NEs, there exists the notion of correlated NE, where you coordinate your action using an external randomized signal