An Introduction to Game Theory
Part I: Strategic Games

Bernhard Nebel

Strategic Game

- A strategic game G consists of
  - a finite set N (the set of players)
  - for each player $i \in N$ a non-empty set A_i (the set of actions or strategies available to player i), whereby $A = \Pi_i A_i$
  - for each player $i \in N$ a function $u_i : A \rightarrow R$ (the utility or payoff function)
  - $G = (N, (A_i), (u_i))$
- If A is finite, then we say that the game is *finite*

Playing the Game

- Each player i makes a decision which action to play: a_i
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
- Then each player gets the payoff $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- Note: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed

Notation

- For *2-player games*, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns
- The payoff for every action profile is specified as a pair x,y, whereby x is the value for player 1 and y is the value for player 2
- Example: For (T,R), player 1 gets x_{12}, and player 2 gets y_{12}

	Player 2 **L** action	Player 2 **R** action
Player1 **T** action	x_{11},y_{11}	x_{12},y_{12}
Player1 **B** action	x_{21},y_{21}	x_{22},y_{22}

Example Game: Bach and Stravinsky

- Two people want to out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

	Bach	Stra-vinsky
Bach	2,1	0,0
Stra-vinsky	0,0	1,2

Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
- This game is also called *chicken.*

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Example Game: Prisoner's Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

	Don't confess	Confess
Don't confess	3,3	0,4
Confess	4,0	1,1

Solving a Game

- What is the right move?
- Different possible solution concepts
  - Elimination of strictly or weakly dominated strategies
  - Maximin strategies (for minimizing the loss in zero-sum games)
  - Nash equilibrium
- How difficult is it to compute a solution?
- Are there always solutions?
- Are the solutions unique?

Strictly Dominated Strategies

- Notation:
  - Let $a = (a_i)$ be a strategy profile
  - $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
  - $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots a_n)$
- Strictly dominated strategy:
  - An strategy $a_j^* \in A_j$ is *strictly dominated* if there exists a strategy a_j' such that for all strategy profiles $a \in A$:

$$u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*)$$

- Of course, it is not rational to play strictly dominated strategies

Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game
- This can be done iteratively
- If this converges to a single strategy profile, the result is unique
- This can be regarded as the result of the game, because it is the only rational outcome

Iterated Elimination: Example

- Eliminate:
  - b4, dominated by b3
  - a4, dominated by a1
  - b3, dominated by b2
  - a1, dominated by a2
  - b1, dominated by b2
  - a3, dominated by a2

➢ Result: (a2,b2)

	b1	b2	b3	b4
a1	1,7	2,5	7,2	0,1
a2	5,2	3,3	5,2	0,1
a3	7,0	2,5	0,4	0,1
a4	0,0	0, 2	0,0	9,1

Iterated Elimination: Prisoner's Dilemma

- Player 1 reasons that "not confessing" is strictly dominated and eliminates this option
- Player 2 reasons that player 1 will not consider "not confessing". So he will eliminate this option for himself as well
- So, they both confess

	Don't confess	Confess
Don't confess	3,3	0,4
Confess	4,0	1,1

Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy $a_j^* \in A_j$ is *weakly dominated* if there exists a strategy a_j' such that for all strategy profiles $a \in A$:

$$u_j(a_{-j}, a_j') \geq u_j(a_{-j}, a_j^*)$$

and for at least one profile $a \in A$:

$$u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*).$$

Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate
    - T (≤M)
    - L (≤R)
    - ➢ Result: (1,1)
  - Eliminate:
    - B (≤M)
    - R (≤L)
    - ➢ Result (2,1)

	L	R
T	2,1	0,0
M	2,1	1,1
B	0,0	1,1

Analysis of the *Guessing 2/3 of the Average* Game

- All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!
- This means, that all strategies above

  2/3 x 67

  can be eliminated
- … and so on
- … until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!

Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?

➢ Nash equilibrium

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Nash Equilibrium

- A *Nash equilibrium* is an action profile $a^* \in A$ with the property that for all players $i \in N$:

  $u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \; \forall \, a_i \in A_i$
- In words, it is an action profile such that there is no incentive for any agent to deviate from it
- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept
- If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium

Example Nash-Equilibrium: Prisoner's Dilemma

- Don't – Don't
  - not a NE
- Don't – Confess (and vice versa)
  - not a NE
- Confess – Confess
  - NE

	Don't confess	Confess
Don't confess	3,3	0,4
Confess	4,0	1,1

Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:
  - not a NE
- Hawk-Hawk
  - not a NE
- Dove-Hawk
  - is a NE
- Hawk-Dove
  - is, of course, another NE
- So, NEs are not necessarily unique

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Auctions

- An object is to be assigned to a player in the set {1,…,n} in exchange for a payment.
- Players *i* valuation of the object is v_i, and $v_1 > v_2 > \ldots > v_n$.
- The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers)
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
- The payment for a *first price* auction is the highest bid.
- What are the Nash equilibria in this case?

Formalization

- Game G = ({$1,\ldots,n$}, (A_i), (u_i))
- A_i: bids $b_i \in$ R$^+$
- $u_i(b_{-i}, b_i) = v_i - b_i$ if i has won the auction, 0 othwerwise
- Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.

Nash Equilibria for First-Price Sealed-Bid Auctions

- The Nash equilibria of this game are all profiles *b* with:
  - $b_i \leq b_1$ for all $i \in \{2, \ldots, n\}$
    - No *i* would bid more than v_2 because it could lead to negative utility
    - If a b_i (with < v_2) is higher than b_1 player 1 could increase its utility by bidding $v_2 + \varepsilon$
    - So 1 wins in all NEs
  - $v_1 \geq b_1 \geq v_2$
    - Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - $b_j = b_1$ for at least one $j \in \{2, \ldots, n\}$
    - Otherwise player 1 could have gotten the object for a lower bid

Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game
- No NE at all! What shall we do here?

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Conclusions

- Strategic games are one-shot games, where everybody plays its move simultaneously
- The game outcome is the action profile resulting from the individual choices.
- Each player gets a payoff based on its payoff function and the resulting action profile.
- Iterated elimination of strictly dominated strategies is a convincing solution concept, but unfortunately, most of the time it does not yield a unique solution
- Nash equilibrium is another solution concept: Action profiles, where no player has an incentive to deviate
- It also might not be unique and there can be even infinitely many NEs.
- Also, there is no guarantee for the existence of a NE