An Introduction to Game Theory
Part I: Strategic Games

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Strategic Game

• A strategic game $G$ consists of
  – a finite set $N$ (the set of players)
  – for each player $i \in N$ a non-empty set $A_i$ (the set of actions or strategies available to player $i$), whereby $A = \prod_i A_i$
  – for each player $i \in N$ a function $u_i : A \rightarrow R$ (the utility or payoff function)
  – $G = (N, (A_i), (u_i))$

• If $A$ is finite, then we say that the game is finite
Playing the Game

- Each player $i$ makes a decision which action to play: $a_i$
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
- Then each player gets the payoff $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- **Note:** While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed
Notation

- For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns.
- The payoff for every action profile is specified as a pair $x,y$, whereby $x$ is the value for player 1 and $y$ is the value for player 2.
- Example: For (T,R), player 1 gets $x_{12}$, and player 2 gets $y_{12}$. 

<table>
<thead>
<tr>
<th></th>
<th>Player 2 L action</th>
<th>Player 2 R action</th>
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<tbody>
<tr>
<td>Player1 T action</td>
<td>$x_{11},y_{11}$</td>
<td>$x_{12},y_{12}$</td>
</tr>
<tr>
<td>Player1 B action</td>
<td>$x_{21},y_{21}$</td>
<td>$x_{22},y_{22}$</td>
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Example Game: Bach and Stravinsky

- Two people want to go out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
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<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
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<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
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Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk.
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove.
- This game is also called chicken.

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<td>0,0</td>
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Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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<table>
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Solving a Game

• What is the right move?
• Different possible solution concepts
  – Elimination of strictly or weakly dominated strategies
  – Maximin strategies (for minimizing the loss in zero-sum games)
  – Nash equilibrium
• How difficult is it to compute a solution?
• Are there always solutions?
• Are the solutions unique?
Strictly Dominated Strategies

- **Notation:**
  - Let \( a = (a_i) \) be a strategy profile
  - \( a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \)
  - \((a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_n)\)

- **Strictly dominated strategy:**
  - An strategy \( a_j^* \in A_j \) is *strictly dominated* if there exists a strategy \( a'_j \) such that for all strategy profiles \( a \in A \):
    \[
    u_j(a_{-j}, a'_j) > u_j(a_{-j}, a_j^*)
    \]

- Of course, it is *not rational* to play strictly dominated strategies
Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game.
- This can be done iteratively.
- If this converges to a single strategy profile, the result is unique.
- This can be regarded as the result of the game, because it is the only rational outcome.
Iterated Elimination: Example

- Eliminate:
  - b4, dominated by b3
  - a4, dominated by a1
  - b3, dominated by b2
  - a1, dominated by a2
  - b1, dominated by b2
  - a3, dominated by a2

Result: (a2, b2)
Iterated Elimination: Prisoner’s Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option.
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well.
- So, they both confess.

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Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy $a_j^* \in A_j$ is *weakly dominated* if there exists a strategy $a_j'$ such that for all strategy profiles $a \in A$:
    $$ u_j(a_{-j}, a_j) \geq u_j(a_{-j}, a_j^*) $$
    and for at least one profile $a \in A$:
    $$ u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*). $$
Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique

**Example:**
- Eliminate
  - $T \ (\leq M)$
  - $L \ (\leq R)$
    - Result: $(1,1)$
  - Eliminate:
    - $B \ (\leq M)$
    - $R \ (\leq L)$
      - Result $(2,1)$

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</tr>
<tr>
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<td>1</td>
</tr>
<tr>
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Diagram:
- **T** 
- **M** 
- **B** 
- **L** 
- **R**
Analysis of the Guessing 2/3 of the Average Game

- All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!
- This means, that all strategies above $\frac{2}{3} \times 67$
  
  can be eliminated
- … and so on
- … until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!
Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?
  - Nash equilibrium

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A **Nash equilibrium** is an action profile $a^* \in A$ with the property that for all players $i \in N$:

$$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \ \forall \ a_i \in A_i$$

In words, it is an action profile such that there is no incentive for any agent to deviate from it.

While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept.

If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium.
Example Nash-Equilibrium: Prisoner’s Dilemma

- Don’t – Don’t
  - not a NE
- Don’t – Confess (and vice versa)
  - not a NE
- Confess – Confess
  - NE

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Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove: not a NE
- Hawk-Hawk: not a NE
- Dove-Hawk: is a NE
- Hawk-Dove: is, of course, another NE
- So, NEs are not necessarily unique

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Auctions

- An **object** is to be **assigned** to a player in the set \( \{1,\ldots,n\} \) in exchange for a payment.
- Players' **valuation** of the object is \( v_i \), and \( v_1 > v_2 > \ldots > v_n \).
- The mechanism to assign the object is a **sealed-bid auction**: the players simultaneously submit bids (non-negative real numbers)
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
- The payment for a **first price auction** is the highest bid.
- What are the Nash equilibria in this case?
Formalization

• Game $G = (\{1,\ldots,n\}, (A_i), (u_i))$

• $A_i$: bids $b_i \in \mathbb{R}^+$

• $u_i(b_{-i}, b_i) = v_i - b_i$ if $i$ has won the auction, 0 otherwise

• Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.
Nash Equilibria for First-Price Sealed-Bid Auctions

The Nash equilibria of this game are all profiles \( b \) with:

- \( b_i \leq b_1 \) for all \( i \in \{2, \ldots, n\} \)
  - No \( i \) would bid more than \( v_2 \) because it could lead to negative utility
  - If a \( b_i \) (with \( < v_2 \)) is higher than \( b_1 \), player 1 could increase its utility by bidding \( v_2 + \epsilon \)
  - So 1 wins in all NEs

- \( v_1 \geq b_1 \geq v_2 \)
  - Otherwise, player 1 either loses the bid (and could increase its utility by bidding more) or would have itself negative utility

- \( b_j = b_1 \) for at least one \( j \in \{2, \ldots, n\} \)
  - Otherwise player 1 could have gotten the object for a lower bid
Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game.
- No NE at all! What shall we do here?

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<tbody>
<tr>
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<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>Tail</td>
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<td>1,-1</td>
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Conclusions

- **Strategic games** are one-shot games, where everybody plays its move simultaneously.
- The game outcome is the **action profile** resulting from the individual choices.
- Each player gets a payoff based on its **payoff function** and the resulting action profile.
- **Iterated elimination of strictly dominated strategies** is a convincing solution concept, but unfortunately, most of the time it does not yield a unique solution.
- **Nash equilibrium** is another solution concept: Action profiles, where no player has an incentive to deviate.
- It also might **not be unique** and there can be even infinitely many NEs.
- Also, there is no guarantee for the existence of a NE.