Mainly based on F. V. Jensen, „Bayesian Networks and Decision Graphs“, Springer-Verlag New York, 2001.

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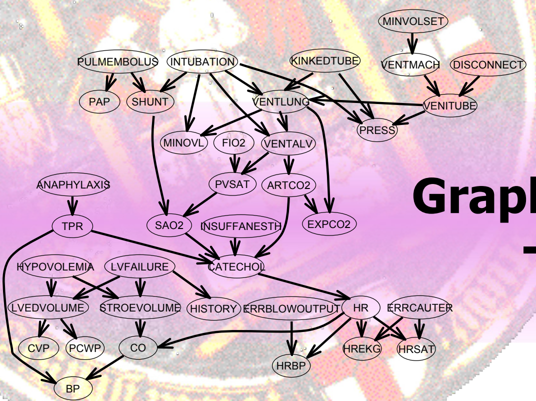

Graphical Models - Inference -

Variable Elimination

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Outline

- Introduction
- Reminder: Probability theory
- Basics of Bayesian Networks
- Modeling Bayesian networks
- Inference
- Excourse: Markov Networks
- Learning Bayesian networks
- Relational Models

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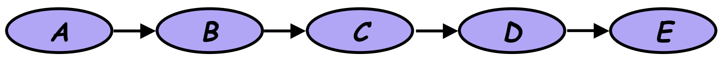
Elimination in Chains

- Forward pass



- Using definition of probability, we have

$$P(e) = \sum_d \sum_c \sum_b \sum_a P(a,b,c,d,e)$$

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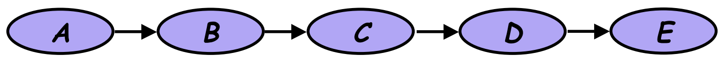
Elimination in Chains

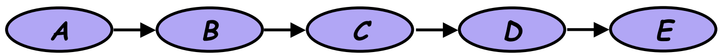


- By chain decomposition, we get

$$P(e) = \sum_d \sum_c \sum_b \sum_a P(a,b,c,d,e)$$

$$P(e) = \sum_d \sum_c \sum_b \sum_a P(a)P(b|a)P(c|b)P(d|c)P(e|d)$$

Elimination in Chains



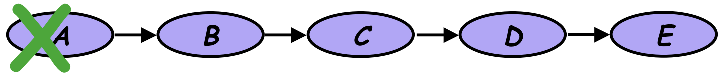
- Rearranging terms ...

$$P(e) = \sum_d \sum_c \sum_b \sum_a P(a)P(b|a)P(c|b)P(d|c)P(e|d)$$

$$P(e) = \sum_d \sum_c \sum_b P(c|b)P(d|c)P(e|d) \sum_a P(a)P(b|a)$$

Elimination in Chains

A has been eliminated



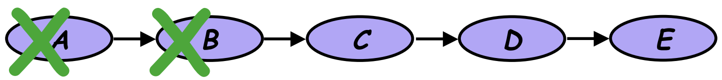
- Now we can perform innermost summation

$$P(e) = \sum_d \sum_c \sum_b P(c|b)P(d|c)P(e|d) \sum_a P(a)P(b|a)$$

$$P(e) = \sum_d \sum_c \sum_b P(c|b)P(d|c)P(e|d)P(b)$$

Elimination in Chains

B has been eliminated



- Rearranging and then summing again, we get

$$P(e) = \sum_d \sum_c \sum_b P(c|b)P(d|c)P(e|d)P(b)$$

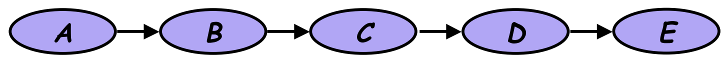
$$P(e) = \sum_d \sum_c P(d|c)P(e|d) \sum_b P(c|b)P(b)$$

$$P(e) = \sum_d \sum_c P(d|c)P(e|d)P(c)$$

...

Elimination in Chains with Evidence

- Similar due to Bayes' Rule $P(a|e) = \frac{P(a,e)}{P(e)}$



- We write the query in explicit form

$$P(a,e) = \sum_d \sum_c \sum_b P(a,b,c,d,e)$$

$$P(a,e) = \sum_d \sum_c \sum_b P(a)P(b|a)P(c|b)P(d|c)P(e|d)$$

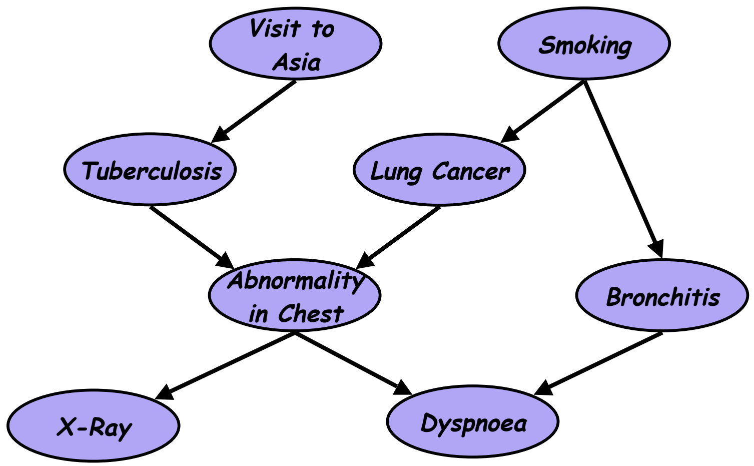
Variable Elimination

- Write query in the form

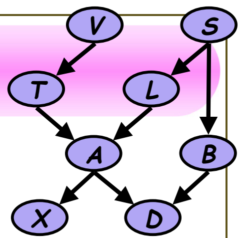
$$P(x_n, \mathbf{e}) = \underbrace{\sum_{x_{j_1}} \sum_{x_{j_2}} \ldots \sum_{x_{j_k}}}_{\text{Eliminate irrelevant variables}} \underbrace{\prod_{i=1}^{n} P(x_i \mid \mathrm{pa}(x_i))}_{\text{Semantic Bayesian Network}}$$

- Iteratively
  - Move all irrelevant terms outside of innermost sum
  - Perform innermost sum, getting a new term
  - Insert the new term into the product

Asia - A More Complex Example



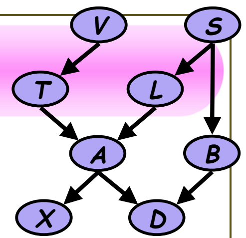
We want to compute *P(d)*
Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a), P(d|a,b)$$

We want to compute *P(d)*
Need to eliminate: *v,s,x,t,l,a,b*



Initial factors

$$\underline{P(v)}P(s)\underline{P(t|v)}P(l|s)P(b|s)P(a|t,l)P(x|a), P(d|a,b)$$

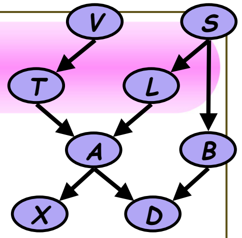
Eliminate: *v*

Compute: $f_v(t) = \sum_v P(v)P(t|v)$

$$\Rightarrow \underline{f_v(t)}P(s)P(l|s)P(b|s)P(a|t,l)P(x|a), P(d|a,b)$$

> $f_v(t) = P(t)$
> but in general the result of eliminating a variable is not necessarily a probability term !

We want to compute *P(d)*
Need to elimininate: *s,x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

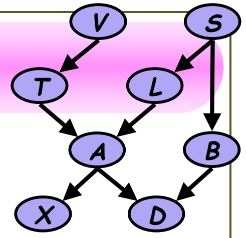
Eliminate: *s*

Compute: $f_s(b,l) = \sum_s P(s)P(l|s)P(b|s)$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a),P(d|a,b)$$

Summing on *s* results in a factor with two arguments $f_s(b,l)$. In general, result of elimination may be a function of several variables

We want to compute *P(d)*
Need to elimininate: *x,t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a),P(d|a,b)$$

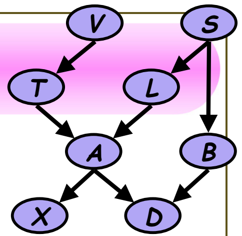
Eliminate: *x*

Compute: $f_x(a) = \sum_x P(x|a)$

$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a|t,l)P(d|a,b)$$

Note that $f_x(a)=1$ for all values of a

We want to compute *P(d)*
Need to elimininate: *t,l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a),P(d|a,b)$$

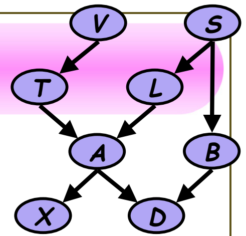
$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a|t,l)P(d|a,b)$$

Eliminate: *t*

Compute: $f_t(a,l) = \sum_t f_v(t)P(a|t,l)$

$$\Rightarrow f_s(b,l)f_x(a)f_t(a,l)P(d|a,b)$$

We want to compute *P(d)*
Need to elimininate: *l,a,b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a),P(d|a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a|t,l)P(d|a,b)$$

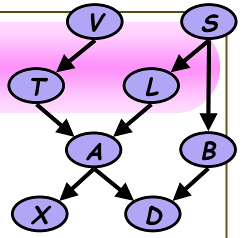
$$\Rightarrow f_s(b,l)f_x(a)f_t(a,l)P(d|a,b)$$

Eliminate: *l*

Compute: $f_l(b,a) = \sum_l f_s(b,l)f_t(a,l)$

$$\Rightarrow f_x(a)f_l(b,a)P(d|a,b)$$

We want to compute *P(d)*
Need to eliminate: *a*,*b*



Initial factors

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$
$$\Rightarrow f_v(t)P(s)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$
$$\Rightarrow f_v(t)f_s(b,l)P(a|t,l)P(x|a),P(d|a,b)$$
$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a|t,l)P(d|a,b)$$
$$\Rightarrow f_s(b,l)f_x(a)f_t(a,l)P(d|a,b)$$
$$\Rightarrow f_x(a)f_l(b,a)P(d|a,b)$$

$$P(d) = f_b(d)$$

Eliminate: *a*,*b*

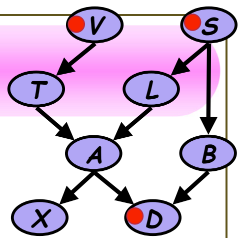
Compute: $f_a(b,d) = \sum_a f_x(a)f_l(b,a)P(d|a,b)$

$$f_b(d) = \sum_b f_a(b,d)$$

Variable Elimination (VE)

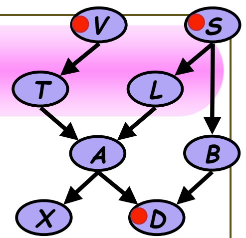
- We now understand **VE** as a sequence of **rewriting operations**
- Actual computation is done in **elimination step**
- **Exactly the same** procedure applies to **Markov networks**
- Computation **depends on order of elimination**

Dealing with Evidence



- How do we deal with evidence?
- Suppose get evidence *V = t, S = f, D = t*
- We want to compute *P(L, V = t, S = f, D = t)*

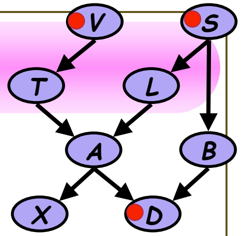
Dealing with Evidence



- We start by writing the factors:

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

Dealing with Evidence



- We start by writing the factors:

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

- **Since we know that *V = t*, we don't need to eliminate *V***
- **Instead, we can replace the factors *P(V)* and *P(T|V)* with** $f_{P(v)} = P(v=t)$ $\quad f_{P(t|v)}(t) = P(t|v=t)$

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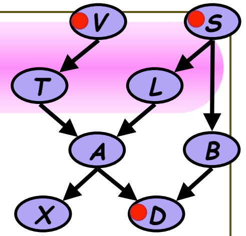
Dealing with Evidence



- We start by writing the factors:

$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a),P(d|a,b)$$

- **Since we know that *V = t*, we don't need to eliminate *V***
- **Instead, we can replace the factors *P(V)* and *P(T|V)* with** $f_{P(v)} = P(v=t)$ $\quad f_{P(t|v)}(t) = P(t|v=t)$
- This "selects" the appropriate parts of the original factors given the evidence
- Note that $f_{P(v)}$ is a constant, and thus does not appear in elimination of other variables

Bayesian Networks - Inference (Variable Elimination)

Given evidence *V = t, S = f, D = t*, **compute** *P(L, V = t, S = f, D = t)*

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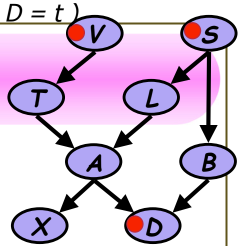
Dealing with Evidence



- Initial factors, after setting evidence

$$f_{P(v)}f_{P(s)}f_{P(t|v)}(t)f_{P(l|s)}(l)f_{P(b|s)}(b)P(a|t,l)P(x|a)f_{P(d|a,b)}(a,b)$$

Bayesian Networks - Inference (Variable Elimination)

Given evidence *V = t, S = f, D = t*, **compute** *P(L, V = t, S = f, D = t)*

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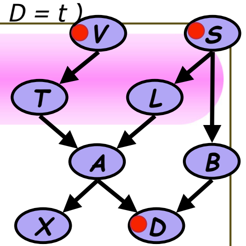
Dealing with Evidence



- Initial factors, after setting evidence

$$f_{P(v)}f_{P(s)}f_{P(t|v)}(t)f_{P(l|s)}(l)f_{P(b|s)}(b)P(a|t,l)P(x|a)f_{P(d|a,b)}(a,b)$$

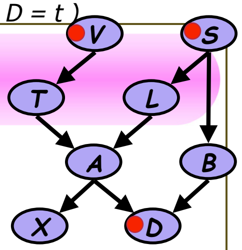
- Eliminating *x*, we get

$$f_{P(v)}f_{P(s)}f_{P(t|v)}(t)f_{P(l|s)}(l)f_{P(b|s)}(b)P(a|t,l)f_x(a)f_{P(d|a,b)}(a,b)$$

Bayesian Networks - Inference (Variable Elimination)

Given evidence *V = t, S = f, D = t,* **compute** *P(L, V = t, S = f, D = t)*

Dealing with Evidence



- Initial factors, after setting evidence

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) P(x|a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, we get

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating t, we get

$$f_{P(v)} f_{P(s)} f_t(a,l) f_{P(l|s)}(l) f_{P(b|s)}(b) f_x(a) f_{P(d|a,b)}(a,b)$$

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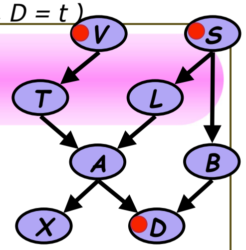
Given evidence *V = t, S = f, D = t,* **compute** *P(L, V = t, S = f, D = t)*

Dealing with Evidence



- Initial factors, after setting evidence

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) P(x|a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, we get

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating t, we get

$$f_{P(v)} f_{P(s)} f_t(a,l) f_{P(l|s)}(l) f_{P(b|s)}(b) f_x(a) f_{P(d|a,b)}(a,b)$$

- Eliminating a, we get

$$f_{P(v)} f_{P(s)} f_a(b,l) f_{P(l|s)}(l) f_{P(b|s)}(b)$$

Bayesian Networks - Inference (Variable Elimination)

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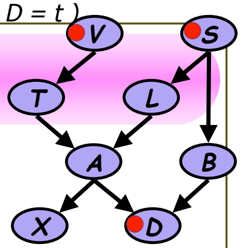
Given evidence *V = t, S = f, D = t,* **compute** *P(L, V = t, S = f, D = t)*

Dealing with Evidence



- Initial factors, after setting evidence

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) P(x|a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, t, a, we get

$$f_{P(v)} f_{P(s)} f_a(b,l) f_{P(l|s)}(l) f_{P(b|s)}(b)$$

- Finally, when eliminating b, we get

Bayesian Networks - Inference (Variable Elimination)

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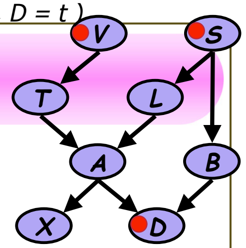
Given evidence *V = t, S = f, D = t,* **compute** *P(L, V = t, S = f, D = t)*

Dealing with Evidence



- Initial factors, after setting evidence

$$f_{P(v)} f_{P(s)} f_{P(t|v)}(t) f_{P(l|s)}(l) f_{P(b|s)}(b) P(a|t,l) P(x|a) f_{P(d|a,b)}(a,b)$$

- Eliminating x, t, a, we get

$$f_{P(v)} f_{P(s)} f_a(b,l) f_{P(l|s)}(l) f_{P(b|s)}(b)$$

- Finally, when eliminating b, we get

$$f_{P(v)} f_{P(s)} f_b(l) f_{P(l|s)}(l)$$

$$P(L, V=t, S=f, D=t) = f_{P(v)} f_{P(s)} f_b(l) f_{P(l|s)}(l)$$

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- Inference (VE, Junction Tree)
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