Classical Logic

Predicate Logic

Knowledge Representation and Reasoning

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Outline

Motivation

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Literature
Why First-Order Logic (FOL)?

- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - All CS students know formal logic
  - Peter is a CS student
  - Therefore, Peter knows formal logic
  - Not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.
Syntax

- **variable symbols**: \( x, y, z, \ldots \)
- **\( n \)-ary function symbols**: \( f, g, \ldots \)
- **constant symbols**: \( a, b, c, \ldots \)
- **\( n \)-ary predicate symbols**: \( P, Q, \ldots \)
- **logical symbols**: \( \forall, \exists, =, \neg, \land, \ldots \)

Terms

\[
\begin{align*}
  t & \longrightarrow x & \text{variable} \\
  & \mid f(t_1, \ldots, t_n) & \text{function application} \\
  & \mid a & \text{constant}
\end{align*}
\]

Formulae

\[
\begin{align*}
  \varphi & \longrightarrow P(t_1, \ldots, t_n) & \text{atomic formula} \\
  & \mid t = t' & \text{identity formulae} \\
  & \mid \ldots & \text{propositional connectives} \\
  & \mid \forall x(\varphi') & \text{universal quantification} \\
  & \mid \exists x(\varphi') & \text{existential quantification}
\end{align*}
\]

**ground term, etc.**: term, etc. without variable occurrences
Semantics: Idea

- In FOL, the universe of discourse consists of objects, functions over these objects, and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $I(x)$ we write $x^I$.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt all these universes.
Formal Semantics: Interpretations

Interpretations: \( I = \langle D, \cdot^I \rangle \) with \( D \) being an arbitrary non-empty set and \( I \) being a function which maps

- \( n \)-ary function symbols \( f \) to \( n \)-ary functions \( f^I \in [D^n \rightarrow D] \),
- constant symbols \( a \) to objects \( a^I \in D \), and
- \( n \)-ary predicates \( P \) to \( n \)-ary relations \( P^I \subseteq D^n \).

Interpretation of ground terms:

\[
(f(t_1, \ldots, t_n))^I = f^I(t_1^I, \ldots, t_n^I) \ (\in D)
\]

Truth of ground atoms:

\[
I \models P(t_1, \ldots, t_n) \ \text{iff} \ \langle t_1^I, \ldots, t_n^I \rangle \in P^I
\]
Examples

\[ \mathcal{D} = \{d_1, \ldots, d_n\}, n \geq 2 \quad \mathcal{D} = \{1, 2, 3, \ldots\} \]

\[ a^I = d_1 \quad 1^I = 1 \]

\[ b^I = d_2 \quad 2^I = 2 \]

\[ \text{eye}^I = \{d_1\} \quad : \quad \text{even}^I = \{2, 4, 6, \ldots\} \]

\[ \text{red}^I = \mathcal{D} \quad \text{succ}^I = \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \]

\[ I \models \text{red}(b) \quad I \not\models \text{even}(3) \]

\[ I \not\models \text{eye}(b) \quad I \models \text{even}(\text{succ}(3)) \]
Formal Semantics: Variable Maps

$V$ is the set of variables. Function $\alpha : V \rightarrow D$ is a variable map.
Notation: $\alpha[x/d]$ is identical to $\alpha$ except for $x$ where $\alpha[x/d](x) = d$.

Interpretation of terms under $I, \alpha$:

$$x^I,\alpha = \alpha(x)$$
$$a^I,\alpha = a^I$$
$$f(t_1, \ldots, t_n)^I,\alpha = f^I(t_1^I,\alpha, \ldots, t_n^I,\alpha)$$

Truth of atomic formulae:

$$I, \alpha \models P(t_1, \ldots, t_n) \text{ iff } \langle t_1^I,\alpha, \ldots, t_n^I,\alpha \rangle \in P^I$$

Example (cont.):

$$\alpha = \{x \mapsto d_1, y \mapsto d_2\} \quad I, \alpha \models \text{red}(x) \quad I, \alpha[y/d_1] \models \text{eye}(y)$$
Formal Semantics: Truth

Truth of $\varphi$ by $I$ under $\alpha$ ($I, \alpha \models \varphi$) is defined as follows.

\[
I, \alpha \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1^{I,\alpha}, \ldots, t_n^{I,\alpha} \rangle \in P^I
\]

\[
I, \alpha \models t_1 = t_2 \quad \text{iff} \quad t_1^{I,\alpha} = t_2^{I,\alpha}
\]

\[
I, \alpha \models \neg \varphi \quad \text{iff} \quad I, \alpha \not\models \varphi
\]

\[
I, \alpha \models \varphi \land \psi \quad \text{iff} \quad I, \alpha \models \varphi \text{ and } I, \alpha \models \psi
\]

\[
I, \alpha \models \varphi \lor \psi \quad \text{iff} \quad I, \alpha \models \varphi \text{ or } I, \alpha \models \psi
\]

\[
I, \alpha \models \varphi \rightarrow \psi \quad \text{iff} \quad \text{if } I, \alpha \models \varphi, \text{ then } I, \alpha \models \psi
\]

\[
I, \alpha \models \varphi \leftrightarrow \psi \quad \text{iff} \quad I, \alpha \models \varphi, \text{ iff } I, \alpha \models \psi
\]

\[
I, \alpha \models \forall x \varphi \quad \text{iff} \quad I, \alpha[x/d] \models \varphi \text{ for all } d \in D
\]

\[
I, \alpha \models \exists x \varphi \quad \text{iff} \quad I, \alpha[x/d] \models \varphi \text{ for some } d \in D
\]
Examples

Ω = \{ \text{eye}(a), \text{eye}(b) \}
\{ \forall x (\text{eye}(x) \rightarrow \text{red}(x)) \}.

\mathcal{D} = \{d_1, \ldots, d_n,\} \quad n > 1

a^I = d_1

b^I = d_1

eye^I = \{d_1\}

\text{red}^I = \mathcal{D}

\alpha = \{(x \mapsto d_1), (y \mapsto d_2)\}

Questions:

\begin{align*}
I, \alpha &\models \text{eye}(b) \lor \neg \text{eye}(b) \quad \text{Yes} \\
I, \alpha &\models \text{eye}(x) \rightarrow \\
&\text{eye}(x) \lor \text{eye}(y) \quad \text{Yes} \\
I, \alpha &\models \text{eye}(x) \rightarrow \text{eye}(y) \quad \text{No} \\
I, \alpha &\models \text{eye}(a) \land \text{eye}(b) \quad \text{Yes} \\
I, \alpha &\models \forall x (\text{eye}(x) \rightarrow \text{red}(x)) \quad \text{Yes} \\
I, \alpha &\models \Theta \quad \text{Yes}
\end{align*}
**Terminology**

$I, \alpha$ is a **model** of $\varphi$ iff

$$I, \alpha \models \varphi.$$ 

A formula can be satisfiable, unsatisfiable, falsifiable, valid.
Two formulae $\varphi$ and $\psi$ are **logically equivalent** ($\varphi \equiv \psi$) iff for all $I, \alpha$:

$$I, \alpha \models \varphi \text{ iff } I, \alpha \models \psi.$$ 

Note: $P(x) \neq P(y)$!
Logical Implication is also similar to propositional logic:

$$\Theta \models \varphi \text{ iff for all } I, \alpha \text{ s.t. } I, \alpha \models \Theta \text{ also } I, \alpha \models \varphi.$$
Free and Bound Variables

Variables can be free or bound (by a quantifier) in a formula:

\[
\begin{align*}
\text{free}(x) &= \{x\} \\
\text{free}(f(t_1, \ldots, t_n)) &= \text{free}(t_1) \cup \ldots \cup \text{free}(t_n) \\
\text{free}(t_1 = t_2) &= \text{free}(t_1) \cup \text{free}(t_2) \\
\text{free}(P(t_1, \ldots, t_n)) &= \text{free}(t_1) \cup \ldots \cup \text{free}(t_n) \\
\text{free}(\neg \varphi) &= \text{free}(\varphi) \\
\text{free}(\varphi \ast \psi) &= \text{free}(\varphi) \cup \text{free}(\psi) \\
\text{free}(\exists x \varphi) &= \text{free}(\varphi) - \{x\} \quad \exists = \forall, \exists
\end{align*}
\]

Example: \(\forall x \ (R(\framebox{y}, \framebox{z}) \land \exists y \ (\neg P(y,x) \lor R(y,\framebox{z})))\)

Framed occurrences are free, all others are bound.
Open & Closed Formulae

- Formulae without free variables are called closed formulae or sentences. Formulae with free variables are called open formulae.

- Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of $\forall$ and $\exists$).

- Note that logical equivalence, satisfiability, and entailment are independent from variable maps if we consider only closed formulae.

- For closed formulae, we omit $\alpha$ in connection with $\models$:

  \[ I \models \varphi. \]
Important Theorems

Theorem (Compactness)

Let $\Phi \cup \{\psi\}$ be a set of closed formulae.

(a) $\Phi \models \psi$ iff there exists a finite subset $\Phi' \subset \Phi$ s.t. $\Phi' \models \psi$.

(b) $\Phi$ is satisfiable iff each finite subset $\Phi' \subset \Phi$ is satisfiable.

Theorem (Löwenheim-Skolem)

Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.
Literature

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