An Introduction to Game Theory

Part V:
Extensive Games with Perfect Information

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Motivation

• So far, all games consisted of just one simultaneous move by all players
• Often, there is a whole sequence of moves and player can react to the moves of the other players
• Examples:
  – board games
  – card games
  – negotiations
  – interaction in a market
Example: Entry Game

- An *incumbent* faces the possibility of *entry* by a *challenger*. The *challenger* may enter (*in*) or not enter (*out*). If it enters, the *incumbent* may either *give in* or *fight*.

- The payoffs are
  - challenger: 1, incumbent: 2 if challenger does not enter
  - challenger: 2, incumbent: 1 if challenger enters and incumbent gives in
  - challenger: 0, incumbent: 0 if challenger enters and incumbent fights

(similar to chicken – but here we have a sequence of moves!)
Formalization: Histories

- The possible developments of a game can be described by a *game tree* or a mechanism to construct a game tree.
- Equivalently, we can use the set of paths starting at the root: all potential *histories* of moves
  - potentially infinitely many (infinite branching)
  - potentially infinitely long
An extensive games with perfect information consists of

- a non-empty, finite set of players $N = \{1, \ldots, n\}$
- a set $H$ (histories) of sequences such that
  - $\langle \rangle \in H$
  - $H$ is prefix-closed
  - if for an infinite sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ every prefix of this sequence is in $H$, then the infinite sequence is also in $H$
  - sequences that are not a proper prefix of another strategy are called terminal histories and are denoted by $Z$. The elements in the sequences are called actions.
- a player function $P: H \setminus Z \to N$,
- for each player $i$ a payoff function $u_i: Z \to \mathbb{R}$

- A game is finite if $H$ is finite
- A game as a finite horizon, if there exists a finite upper bound for the length of histories
Entry Game – Formally

• players N = {1,2} (1: challenger, 2: incumbent)
• histories H = {⟨⟩, ⟨out⟩, ⟨in⟩, ⟨in, fight⟩, ⟨in, give_in⟩}
• terminal histories: Z = {⟨out⟩, in, fight⟩, ⟨in, give_in⟩}
• player function:
  – P(⟨⟩) = 1
  – P(⟨in⟩) = 2
• payoff function
  – u₁(⟨out⟩)=1, u₂(⟨out⟩)=2
  – u₁(⟨in, fight⟩)=0, u₂(⟨in, fight⟩)=0
  – u₁(⟨in,give_in⟩)=2, u₂(⟨in,give_in⟩)=1
Strategies

- The number of possible actions after history $h$ is denoted by $A(h)$.
- A strategy for player $i$ is a function $s_i$ that maps each history $h$ with $P(h) = i$ to an element of $A(h)$.
- Notation: Write strategy as a sequence of actions as they are to be chosen at each point when visiting the nodes in the game tree in breadth-first manner.

- Possible strategies for player 1:
  - AE, AF, BE, BF
- for player 2:
  - C, D
- Note: Also decisions for histories that cannot happen given earlier decisions!
Outcomes

- The outcome $O(s)$ of a strategy profile $s$ is the terminal history that results from applying the strategies successively to the histories starting with the empty one.
- What is the outcome for the following strategy profiles?
  - $O(AF,C) =$
  - $O(AF,D) =$
  - $O(BF,C) =$
Nash Equilibria in Extensive Games with Perfect Information

• A strategy profile $s^*$ is a Nash Equilibrium in an extensive game with perfect information if for all players $i$ and all strategies $s_i$ of player $i$:

$$u_i(O(s^*_{-i}, s^*_i)) \geq u_i(O(s^*_{-i}, s_i))$$

• Equivalently, we could define the strategic form of an extensive game and then use the existing notion of Nash equilibrium for strategic games.
The Entry Game - again

- Nash equilibria?
  - In, Give in
  - Out, Fight
- But why should the challenger take the “threat” seriously that the incumbent starts a fight?
- Once the challenger has played “in”, there is no point for the incumbent to reply with “fight”. So “fight” can be regarded as an empty threat

\[
\begin{array}{c|cc}
& \text{Give in} & \text{Fight} \\
\hline
\text{In} & 2,1 & 0,0 \\
\text{Out} & 1,2 & 1,2 \\
\end{array}
\]

- Apparently, the Nash equilibrium out, fight is not a real “steady state” – we have ignored the sequential nature of the game
Sub-games

• Let $G=(N,H,P,(u_i))$ be an extensive game with perfect information. For any non-terminal history $h$, the sub-game $G(h)$ following history $h$ is the following game: $G’=(N,H’,P’,(u_i’))$ such that:
  – $H’$ is the set of histories such that for all $h’$: $(h,h’) \in H$
  – $P’(h’) = P((h,h’))$
  – $u_i’(h’) = u_i((h,h’))$

*How many sub-games are there?*
Applying Strategies to Sub-games

- If we have a strategy profile $s^*$ for the game $G$ and $h$ is a history in $G$, then $s^*_h$ is the strategy profile after history $h$, i.e., it is a strategy profile for $G(h)$ derived from $s^*$ by considering only the histories following $h$.
- For example, let $((\text{out}), (\text{fight}))$ be a strategy profile for the entry game. Then $((\text{}), (\text{fight}))$ is the strategy profile for the sub-game after player 1 played “in”.
Sub-game Perfect Equilibria

• A sub-game perfect equilibrium (SPE) of an extensive game with perfect information is a strategy profile $s^*$ such that for all histories $h$, the strategies in $s^*_h$ are optimal for all players.

• Note: $((\text{out}), (\text{fight}))$ is not a SPE!

• Note: A SPE could also be defined as a strategy profile that induces a NE in every sub-game
Example: Distribution Game

- Two objects of the same kind shall be distributed to two players. Player 1 suggest a distribution, player 2 can accept (+) or reject (-). If she accepts, the objects are distributed as suggested by player 1. Otherwise nobody gets anything.

- NEs?

- SPEs?

- ((2,0),+xx) are NEs
- ((2,0),--x) are NEs
- ((1,1),-+x) are NEs
- ((0,1),--+) is a NE

Only
- ((2,0),+++) is a SPE
- ((1,1),-++) is a SPE
Existence of SPEs

• **Infinite games** may not have a SPE
  – Consider the 1-player game with actions \([0,1)\) and payoff \(u_1(a) = a\).

• If a game **does not have a finite horizon**, then it may not possess an SPE:
  – Consider the 1-player game with infinite histories such that the infinite histories get a payoff of 0 and all finite prefixes extended by a termination action get a payoff that is proportional to their length.
Finite Games Always Have a SPE

- Length of a sub-game = length of longest history
- Use **backward induction**
  - Find the optimal play for all sub-games of length 1
  - Then find the optimal play for all sub-games of length 2 (by using the above results)
  - ....
  - until length n = length of game
    - game has an SPE
- SPE is not necessarily unique – agent may be indifferent about some outcomes
- All SPEs can be found this way!
Strategies and Plans of Action

- Strategies contain decisions for **unreachable** situations!
- Why should player 1 worry about the choice after A,C if he will play B?
- Can be thought of as
  - what player 2 believes about player 1
  - what will happen if by mistake player 1 chooses A
  - Player 1 actually would play
The Distribution Game - again

• Now it is easy to find all SPEs
• Compute optimal actions for player 2
• Based on the results, consider actions of player 1
Another Example: The Chain Store Game

• If we play the entry game for $k$ periods and add up the payoff from each period, what will be the SPEs?

• By backward induction, the only SPE is the one, where in every period (in, give_in) is selected

• However, for the incumbent, it could be better to play sometimes fight in order to “build up a reputation” of being aggressive.
Yet Another Example: The Centipede Game

- The players move alternately
- Each prefers to stop in his move over the other player stopping in the next move
- However, if it is not stopped in these two periods, this is even better
- What is the SPE?
Relationship to Minimax

• Similarities to \textit{Minimax}
  – solving the game by searching the game tree bottom-up, choosing the optimal move at each node and propagating values upwards

• Differences
  – More than two players are possible in the backward-induction case
  – Not just one number, but an entire payoff profile

• So, is \textit{Minimax} just a special case?
Possible Extensions

• One could add random moves to extensive games. Then there is a special player which chooses its actions randomly
  – SPEs still exist and can be found by backward induction. However, now the expected utility has to be optimized

• One could add simultaneous moves, that the players can sometimes make moves in parallel
  – SPEs might not exist anymore (simple argument!)

• One could add “imperfect information”: The players are not always informed about the moves other players have made.
Conclusions

• Extensive games model games in which more than one simultaneous move is allowed
• The notion of Nash equilibrium has to be refined in order to exclude implausible equilibria – those with empty threats
• Sub-game perfect equilibria capture this notion
• In finite games, SPEs always exist
• All SPEs can be found by using backward induction
• Backward induction can be seen as a generalization of the Minimax algorithm
• A number of plausible extensions are possible: simultaneous moves, random moves, imperfect information