An Introduction to Game Theory
Part IV:
Games with Imperfect Information
Bernhard Nebel
Motivation

• So far, we assumed that all players have perfect knowledge about the preferences (the payoff function) of the other players
• Often unrealistic
• For example, in auctions people are not sure about the valuations of the others – what to do in a sealed bid auction?
Example

• Let’s assume the BoS game, where player 1 is not sure, whether player 2 wants to meet her to avoid her,
• She assumes a probability of 0.5 for each case.
• Player 2 knows the preferences of player 1
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<th>Bach</th>
<th>Stravinsky</th>
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Prob. 0.5

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Prob. 0.5
What is the Payoff?

• Player 1 views player 2 as being one of two possible *types*
• Each of these types may make an independent decision
• So, the friendly player 2 may choose B and the unfriendly one S: (B,S)
• **Expected payoff** when player 1 plays B:
  \[ 0.5 \times 2 + 0.5 \times 0 = 1 \]
Expected Payoffs & Nash Equilibrium

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<tr>
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<th>(B,B)</th>
<th>(B,S)</th>
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<tr>
<td>B</td>
<td>2 (1,0)</td>
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<td>0.5 (0,0)</td>
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• A **Nash equilibrium** in pure strategies is a triple \((x,(y,z))\) of actions such that:
  – the action \(x\) of player 1 is optimal given the actions \((y,z)\) of both types of player 2 and the belief about the state
  – the actions \(y\) and \(z\) of each type of player 2 are optimal given the action \(x\) of player 1
### Nash Equilibria?

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- Is there a Nash equilibrium?  
  - Yes: B, (B,S)
- Is there a NE where player 1 plays S?  
  - No
Formalization:
States and Signals

• There are **states**, which completely determine the preferences / payoff functions
  – In our example: *friendly* and *unfriendly*

• Before the game starts, each player receives a **signal** that tells her something about the state
  – In our example:
    • Player 2 receives a states, which type she is
    • Player 1 gets no information about the state and has only her beliefs about probabilities.

• Although, the actions for non-realized types of player 2 are irrelevant for player 2, they are necessary for player 1 (and therefore also for player 2) when deliberating about possible action profiles and their payoffs
General Bayesian Games

• A Bayesian game consists of
  – a set of players \( N = \{1, \ldots, n\} \)
  – a set of states \( \Omega = \{\omega_1, \ldots, \omega_k\} \)
• and for each player \( i \)
  – a set of actions \( A_i \)
  – a set of signals \( T_i \) and a signal function \( \tau_i : \Omega \to T_i \)
  – for each signal a belief about the possible states (a probability distribution over the states associated with the signal) \( Pr(\omega | t_i) \)
  – a payoff function \( u_i(a, \omega) \) over pairs of action profiles and states, where the expected value for \( a_i \) represents the preferences:
    \[
    \sum_{\omega \in \Omega} Pr(\omega | t_i) u_i((a_i, \hat{a}_i(\omega)), \omega)
    \]
  with \( \hat{a}_i(\omega) \) denoting the choice by \( i \) when she has received the signal \( \tau_i(\omega) \)
Example: BoS with Uncertainty

- Players: \{1, 2\}
- States: \{friendly, unfriendly\}
- Actions: \{B, S\}
- Signals: \( T = \{a, b, c\} \)
  - \( \tau_1(\omega_i) = a \) for \( i = 1, 2 \)
  - \( \tau_2(\text{friendly}) = b, \tau_2(\text{unfriendly}) = c, \)
- Beliefs:
  - \( \Pr(\text{friendly} | a) = 0.5, \Pr(\text{unfriendly} | a) = 0.5 \)
  - \( \Pr(\text{friendly} | b) = 1, \Pr(\text{friendly} | b) = 0 \)
  - \( \Pr(\text{friendly} | c) = 0, \Pr(\text{friendly} | c) = 1 \)
- Payoffs: As in the left and right tables on the slide
Example: Information can hurt

• In single-person games, knowledge can never hurt, but here it can!
• Two players, both don’t know which state und consider both states $\omega_1$ and $\omega_2$ as equally probable (0.5)
• Note: Preferences of player 1 are known, while the preferences of player 2 are unknown (to both!)

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<tr>
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<tr>
<td>$\omega_1$</td>
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<td>T</td>
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Example (cont.)

- Player 2’s unique best response is: L
- For this reason, player 1 will play B
- Payoff: 6,6 – only NE, even when mixed strategies!
- When player 2 can distinguish the states, R and M are dominating actions
- \((T,(R,M))\) is the unique NE

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Incentives and Uncertain Knowledge May Lead to Suboptimal Solutions

- \( \tau_1(\alpha) = a, \ \tau_1(\beta) = b, \ \tau_1(\gamma) = b \)
  - \( Pr(\alpha|a) = 1 \)
  - \( Pr(\beta|b) = 0.75, \ Pr(\gamma|b) = 0.25 \)
- \( \tau_2(\alpha) = c, \ \tau_2(\beta) = c, \ \tau_2(\gamma) = d \)
  - \( Pr(\alpha|c) = 0.75, \ Pr(\beta|c) = 0.25 \)
  - \( Pr(\gamma|d) = 1 \)

- In state \( \gamma \), there are 2 NEs
- In state \( \gamma \), player 2 knows her preferences, but player 1 does not know that!
- The incentive for player 1 to play \( R \) in state \( \alpha \) „infects“ the game and only \((R,R),(R,R)\) is an NE

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The Infection

- Player 1 must play R when receiving signal a (= state $\alpha$)!
- Player 2 will therefore never play L when receiving c (= $\alpha$ or $\beta$)
- For this reason, player 1 will never play L when receiving b (= $\beta$ or $\gamma$)
- Therefore player 2 will also play R when receiving d (= $\gamma$)
- Therefore the unique NE is ((R,R),(R,R))!

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$Pr(\alpha|a) = 1$
$Pr(\beta|b) = 0.75$, $Pr(\gamma|b) = 0.25$

$\tau_2(\alpha) = c$, $\tau_2(\beta) = c$, $\tau_2(\gamma) = d$
$Pr(\alpha|c) = 0.75$, $Pr(\beta|c) = 0.25$
$Pr(\gamma|d) = 1$
Auctions with Imperfect Information

- **Players**: $N = \{1, \ldots, n\}$
- **States**: the set of all profiles of valuations $(v_1, \ldots, v_n)$, where $0 \leq v_i \leq v_{\text{max}}$
- **Actions**: Set of possible bids
- **Signals**: The set of the player $i$'s valuation $\tau_i(v_1, \ldots, v_n) = v_i$
- **Beliefs**: $F(v)$ is the probability that the other bidder values of the object is at most $v$, i.e., $F(v_1)x\ldotsxF(v_{i-1})xF(v_{i+1})x\ldotsxF(v_n)$ is the probability, that all other players $j$ value the object at most $v_j$
- **Payoff**: $u_i(b, (v_1, \ldots, v_n)) = (v_i - P(b))/m$ if $b_j \leq b$ for all $i \neq j$ and $b_j = b$ for $m$ players and $P(b)$ being the **price function**:
  - $P(b)$ the highest bid = first price auction
  - $P(b)$ the second highest bid = second price auction
Private and Common Values

• If the valuations are *private*, that is each one cares only about the his one appreciation (e.g., in art),
  – valuations are completely independent
  – one does not gain information when people submit public bids

• In an auction with *common valuations*, which means that the players share the value system but may be unsure about the real value (antiques, technical devices, exploration rights),
  – valuations are not independent
  – one might gain information from other players bids

• Here we consider private values
Second Price Sealed Bid Auction

- \( P(b) \) is what the second highest bid was
- As in the perfect information case (see exercise):
  - It is a weakly dominating action to bid one's own valuation \( v_i \)
  - There exist other, non-efficient, equilibria
First Price Sealed Bid Auction

- A bid of $v_i$ weakly dominates any bid higher than $v_i$
- A bid of $v_i$ does not weakly dominate a bid lower than $v_i$
- A bid lower than $v_i$ weakly dominates $v_i$
- NE probably at a point below $v_i$
- General analysis is quite involved
- Simplifications:
  - only 2 players
  - $v_{max} = 1$
  - uniform distribution of valuations, i.e., $F(v) = v$
First Price Sealed Bid Auction (2)

- Let $B_i(v)$ the bid of type $v$ for player $i$.
- **Claim**: Under the mentioned conditions, the game has a NE for $B_i(v) = v/2$.
- Assume that player 2 bids this way, then as far as player 1 is concerned, player 2’s bids are uniformly distributed between 0 and 0.5.
- Thus, if player 1 bids $b_1 > 0.5$, she wins. Otherwise, the probability that she wins is $F(2b_1)$.
- The payoff is
  - $v_1 - b_1$ if $b_1 > 0.5$
  - $2b_1 (v_1 - b_1) = 2b_1v_1 - 2b_1^2$ if $0 \leq b_1 \leq 0.5$
First Price Sealed Bid Auction (3)

• In other words, $0.5v_1$ is the best response to $B_2(v) = v/2$ for player 1.
• Since the players are symmetric, this also holds for player 2.
• Hence, this is a NE.
• In general, for $m$ players, the NE is $B_i(v) = v/m$ for $m$ players.
• Can also be shown for general distributions.
Conclusion

- If the players are not fully informed about their own and others utilities, we have imperfect information.
- The technical tool to model this situation are Bayesian games.
- New concepts are states, signals, beliefs and expected utilities over the believed distributions over states.
- Being informed can hurt!
- Auctions are more complicated in the imperfect information case, but can still be solved.