An Introduction to Game Theory
Part I: Strategic Games

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Strategic Game

• A **strategic game** $G$ consists of
  
  – a finite set $N$ (the set of **players**)
  
  – for each player $i \in N$ a non-empty set $A_i$ (the set of **actions** or **strategies** available to player $i$), whereby $A = \prod_i A_i$
  
  – for each player $i \in N$ a function $u_i : A \to \mathbb{R}$ (the **utility** or **payoff** function)

  – $G = (N, (A_i), (u_i))$

• If $A$ is finite, then we say that the game is **finite**
Playing the Game

• Each player $i$ makes a decision which action to play: $a_i$
• All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
• Then each player gets the payoff $u_i(a^*)$
• Of course, each player tries to maximize its own payoff, but what is the right decision?
• **Note:** While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  – If we want to model something like this, the payoff function must be changed
Notation

- For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns.
- The payoff for every action profile is specified as a pair $x,y$, whereby $x$ is the value for player 1 and $y$ is the value for player 2.
- Example: For (T,R), player 1 gets $x_{12}$, and player 2 gets $y_{12}$.
Example Game: Bach and Stravinsky

- Two people want to go together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

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<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
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<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
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<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
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Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk.
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove.
- This game is also called *chicken*.

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<tr>
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<td>Dove</td>
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Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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Solving a Game

• What is the right move?
• Different possible solution concepts
  – Elimination of strictly or weakly dominated strategies
  – Maximin strategies (for minimizing the loss in zero-sum games)
  – Nash equilibrium
• How difficult is it to compute a solution?
• Are there always solutions?
• Are the solutions unique?
Strictly Dominated Strategies

• Notation:
  – Let $a = (a_i)$ be a strategy profile
  – $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
  – $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots a_n)$

• Strictly dominated strategy:
  – An strategy $a_j^* \in A_j$ is strictly dominated if there exists a strategy $a'_j$ such that for all strategy profiles $a \in A$:
    \[ u_j(a_{-j}, a'_j) > u_j(a_{-j}, a_j^*) \]

• Of course, it is not rational to play strictly dominated strategies
Iterated Elimination of Strictly Dominated Strategies

• Since strictly dominated strategies will never be played, one can eliminate them from the game
• This can be done iteratively
• If this converges to a single strategy profile, the result is unique
• This can be regarded as the result of the game, because it is the only rational outcome
Iterated Elimination: Example

• Eliminate:
  – b4, dominated by b3
  – a4, dominated by a1
  – b3, dominated by b2
  – a1, dominated by a2
  – b1, dominated by b2
  – a3, dominated by a2

➢ Result: \((a2, b2)\)
Iterated Elimination: Prisoner’s Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option.
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well.
- So, they both confess.

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Weakly Dominated Strategies

• Instead of strict domination, we can also go for weak domination:
  – An strategy \( a_j^* \in A_j \) is **weakly dominated** if there exists a strategy \( a_j' \) such that for all strategy profiles \( a \in A \):

    \[ u_j(a_{-j}, a_j') \geq u_j(a_{-j}, a_j^*) \]

    and for at least one profile \( a \in A \):

    \[ u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*). \]
Results of Iterative Elimination of Weakly Dominated Strategies

• The result is not necessarily unique

• Example:
  – Eliminate
    • T (≤M)
    • L (≤R)
      ➢ Result: (1,1)
  – Eliminate:
    • B (≤M)
    • R (≤L)
      ➢ Result (2,1)
Analysis of the Guessing 2/3 of the Average Game

• All strategies above 67 are weakly dominated, since they will never ever lead to winning the prize, so they can be eliminated!
• This means, that all strategies above \[\frac{2}{3} \times 67\] can be eliminated
• … and so on
• … until all strategies above 1 have been eliminated!
• So: The rationale strategy would be to play 1!
Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept.
- Unfortunately, often dominated strategies do not exist.
- What do we do in this case?
  - Nash equilibrium

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Nash Equilibrium

- A *Nash equilibrium* is an action profile $a^* \in A$ with the property that for all players $i \in N$:
  $$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \ \forall \ a_i \in A_i$$

- In words, it is an action profile such that there is no incentive for any agent to deviate from it.

- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept.

- If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium.
Example Nash-Equilibrium: Prisoner’s Dilemma

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- Don’t – Don’t: not a NE
- Don’t – Confess (and vice versa): not a NE
- Confess – Confess: NE
Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove: not a NE
- Hawk-Hawk: not a NE
- Dove-Hawk: is a NE
- Hawk-Dove: is, of course, another NE
- So, NEs are not necessarily unique
Auctions

• An object is to be assigned to a player in the set \( \{1, \ldots, n\} \) in exchange for a payment.
• Players' evaluation of the object is \( v_i \), and \( v_1 > v_2 > \ldots > v_n \).
• The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers)
• The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
• The payment for a first price auction is the highest bid.
• What are the Nash equilibria in this case?
Formalization

• Game G = ({1,...,n}, (A_i), (u_i))
• A_i: bids b_i ∈ \mathbb{R}^+
• u_i(b_{-i}, b_i) = v_i - b_i if i has won the auction, 0 otherwise
• Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.
The Nash equilibria of this game are all profiles $b$ with:

- $b_i \leq b_1$ for all $i \in \{2, \ldots, n\}$
  - No $i$ would bid more than $v_2$ because it could lead to negative utility
  - If a $b_i$ (with $b_i < v_2$) is higher than $b_1$ player 1 could increase its utility by bidding $v_2 + \varepsilon$
  - So 1 wins in all NEs

- $v_1 \geq b_1 \geq v_2$
  - Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility

- $b_j = b_1$ for at least one $j \in \{2, \ldots, n\}$
  - Otherwise player 1 could have gotten the object for a lower bid
Another Game: Matching Pennies

• Each of two people chooses either **Head** or **Tail**. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.

• This is also a **zero-sum** or **strictly competitive** game.

• No NE at all! What shall we do here?

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<tr>
<th></th>
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<th>Tail</th>
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<tr>
<td>Head</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
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Conclusions

• **Strategic games** are one-shot games, where everybody plays its move simultaneously.
• The game outcome is the **action profile** resulting from the individual choices.
• Each player gets a payoff based on its **payoff function** and the resulting action profile.
• **Iterated elimination of strictly dominated strategies** is a convincing solution concept, but unfortunately, most of the time it does not yield a unique solution.
• **Nash equilibrium** is another solution concept: Action profiles, where no player has an incentive to deviate.
• It also might **not be unique** and there can be even infinitely many NEs.
• Also, there is no guarantee for the **existence** of a NE.