# Qualitative Representation and Reasoning

## Allen's Interval Calculus

Knowledge Representation and Reasoning

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# Allen's Interval Calculus – Outline







# Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- NLP – we do not have precise time points
- Planning – we do not want to commit to time points too early
- Scenario descriptions – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- Time points: actions and events are instantaneous, or we consider their beginning and ending
- Time intervals: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?





# Motivation: Example

Consider a planning scenario for multimedia generation:

P1: *Display* Picture1
P2: *Say* "Put the plug in."
P3: *Say* "The device should be shut off."
P4: *Point to* Plug-in-Picture1.

Temporal relations between events:

| | | |
|---|---|---|
| P2 | should happen during | P1 |
| P3 | should happen during | P1 |
| P2 | should happen before or directly precede | P3 |
| P4 | should happen during or end together with | P2 |

⇝ P4 happens before or directly precedes P3"

⇝ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

# Allen's Interval Calculus

- Allen's interval calculus: time intervals and binary relations over them
- *Time intervals*: $X = (X^-, X^+)$, where $X^-$ and $X^+$ are interpreted over the reals and $X^- < X^+$ ($\rightsquigarrow$ naïve approach)
- Relations between concrete intervals, e. g.:

  (1.0,2.0) *strictly before* (3.0,5.5)
  (1.0,3.0) *meets* (3.0,5.5)
  (1.0,4.0) *overlaps* (3.0,5.5)
  ...

$\rightsquigarrow$ Which relations are conceivable?





# The Base Relations

How many ways are there to order the four points of two intervals?

| Relation | Symbol | Name |
|---|---|---|
| $\{(X,Y) : X^- < X^+ < Y^- < Y^+\}$ | $\prec$ | before |
| $\{(X,Y) : X^- < X^+ = Y^- < Y^+\}$ | $\mathtt{m}$ | meets |
| $\{(X,Y) : X^- < Y^- < X^+ < Y^+\}$ | $\mathtt{o}$ | overlaps |
| $\{(X,Y) : X^- = Y^- < X^+ < Y^+\}$ | $\mathtt{s}$ | starts |
| $\{(X,Y) : Y^- < X^- < X^+ = Y^+\}$ | $\mathtt{f}$ | finishes |
| $\{(X,Y) : Y^- < X^- < X^+ < Y^+\}$ | $\mathtt{d}$ | during |
| $\{(X,Y) : Y^- = X^- < X^+ = Y^+\}$ | $\equiv$ | equal |

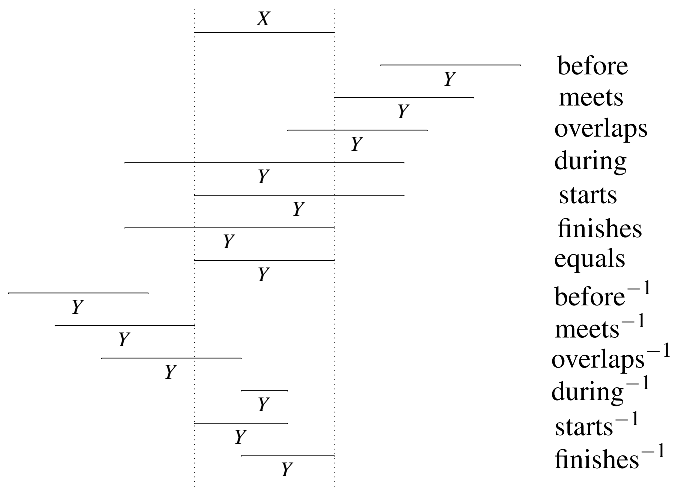
and the **converse** relations (obtained by exchanging $X$ and $Y$)

$\rightsquigarrow$ These relations are JEPD.





# The 13 Base Relations Graphically








# Disjunctive Descriptions

- Assumption: We don't have precise information about the relation between $X$ and $Y$, e. g.:

$$X \mathbin{\mathtt{o}} Y \text{ or } X \mathbin{\mathtt{m}} Y$$

- ...modelled by sets of base relations (meaning the union of the relations):

$$X \{\mathtt{o}, \mathtt{m}\} Y$$

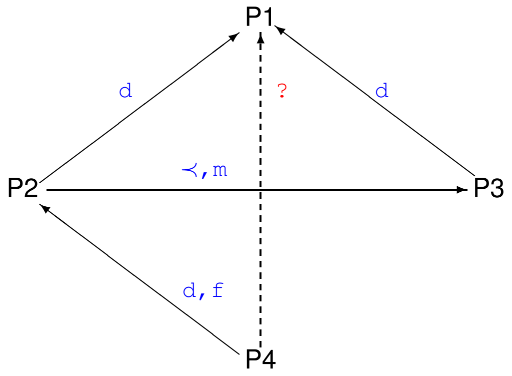
$\rightsquigarrow$ $2^{13}$ imprecise relations (incl. $\emptyset$ and $\mathbf{B}$)

Example of an indefinite qualitative description:

$$\left\{ X \{\mathtt{o}, \mathtt{m}\} Y, Y \{\mathtt{m}\} Z, X \{\mathtt{o}, \mathtt{m}\} Z \right\}$$

# Our Example . . . Formal

P1: *Display* Picture1
P2: *Say* "Put the plug in."
P3: *Say* "The device should be shut off."
P4: *Point to* Plug-in-Picture1.




Compose the constraints: $P4\,\{\mathtt{d},\mathtt{f}\}\,P2$ and $P2\,\{\mathtt{d}\}\,P1$: $P4\,\{\mathtt{d}\}\,P1$.



| | $\prec$ | $\succ$ | d | $d^{-1}$ | o | $o^{-1}$ | m | $m^{-1}$ | s | $s^{-1}$ | f | $f^{-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\prec$ | $\prec$ | **B** | $\prec$ o m d s | $\prec$ | $\prec$ | $\prec$ o m d s | $\prec$ | $\prec$ o m d s | $\prec$ | $\prec$ | $\prec$ o m d s | $\prec$ |
| $\succ$ | **B** | $\succ$ | $\succ$ $o^{-1}$ $m^{-1}$ d f | $\succ$ | $\succ$ $o^{-1}$ $m^{-1}$ d f | $\succ$ | $\succ$ $o^{-1}$ $m^{-1}$ d f | $\succ$ | $\succ$ $o^{-1}$ $m^{-1}$ d f | $\succ$ | $\succ$ | $\succ$ |
| d | $\prec$ | $\succ$ | d | **B** | $\prec$ o m d s | $\succ$ $o^{-1}$ $m^{-1}$ d f | $\prec$ | $\succ$ | d | $\succ$ $o^{-1}$ $m^{-1}$ d f | d | $\prec$ o m d s |
| $d^{-1}$ | $\prec$ o m $d^{-1}$ $f^{-1}$ | $\succ$ $o^{-1}$ $m^{-1}$ $d^{-1}$ $s^{-1}$ | **B**− $\prec\succ$ m $m^{-1}$ | $d^{-1}$ | o $d^{-1}$ $f^{-1}$ | $o^{-1}$ $d^{-1}$ $s^{-1}$ | o $d^{-1}$ $f^{-1}$ | $o^{-1}$ $d^{-1}$ $s^{-1}$ | o $d^{-1}$ $f^{-1}$ | $d^{-1}$ | $o^{-1}$ $d^{-1}$ $s^{-1}$ | $d^{-1}$ |
| o | $\prec$ | $\succ$ $o^{-1}$ $m^{-1}$ $d^{-1}$ $s^{-1}$ | o d s | $\prec$ o m $d^{-1}$ $f^{-1}$ | $\prec$ o m | **B**− $\prec\succ$ m $m^{-1}$ | $\prec$ | $o^{-1}$ $d^{-1}$ $s^{-1}$ | o | $d^{-1}$ $f^{-1}$ o | d s o | $\prec$ o m |
| $o^{-1}$ | $\prec$ o m, $d^{-1}$ $f^{-1}$ | $\succ$ | $o^{-1}$ d f | $\succ$, $o^{-1}$ $m^{-1}$$d^{-1}$ $s^{-1}$ | **B**− $\prec\succ$ m $m^{-1}$ | $\succ$ $o^{-1}$ $m^{-1}$ | o $d^{-1}$ $f^{-1}$ | $\succ$ | $o^{-1}$ d f | $o^{-1}$ $\succ$ $m^{-1}$ | $o^{-1}$ | $o^{-1}$ $d^{-1}$ $s^{-1}$ |
| m | $\prec$ | $\succ$ $o^{-1}$ $m^{-1}$$d^{-1}$ $s^{-1}$ | o d s | $\prec$ | $\prec$ | o d s | $\prec$ | f $f^{-1}$ $\equiv$ | m | m | d s o | $\prec$ |
| $m^{-1}$ | $\prec$ o m $d^{-1}$ $f^{-1}$ | $\succ$ | $o^{-1}$ d f | $\succ$ | $o^{-1}$ d f | $\succ$ | s $s^{-1}$ $\equiv$ | $\succ$ | d f $o^{-1}$ | $\succ$ | $m^{-1}$ | $m^{-1}$ |
| s | $\prec$ | $\succ$ | d | $\prec$ o m $d^{-1}$ $f^{-1}$ | $\prec$ o m | $o^{-1}$ d f | $\prec$ | $m^{-1}$ | s | s $s^{-1}$ $\equiv$ | d | $\prec$ m o |
| $s^{-1}$ | $\prec$ o m $d^{-1}$ $f^{-1}$ | $\succ$ | $o^{-1}$ .d f | $d^{-1}$ | o $d^{-1}$ $f^{-1}$ | $o^{-1}$ | o $d^{-1}$ $f^{-1}$ | $m^{-1}$ | s $s^{-1}$ $\equiv$ | $s^{-1}$ | $o^{-1}$ | $d^{-1}$ |
| f | $\prec$ | $\succ$ | d | $\succ$ $o^{-1}$ $m^{-1}$$d^{-1}$ | o d | $\succ$ $o^{-1}$ | m | $\succ$ | d | $\succ$ $o^{-1}$ | f | f $f^{-1}$ |



# Outlook

- Using the composition table and the rules about operations on relations, we can deduce new relations between time intervals.
- What would be a systematic approach?
- How costly is that?
- Is that complete?
- If not, could it be complete on a subset of the relation system?





# Reasoning in Allen's Interval Calculus

# Constraint Propagation – The Naive Algorithm

Enforcing path-consistency using the straight-forward method:
Let *Table* $[i,j]$ be an array of size $|n| \times |n|$ ($n$: number of intervals), in which we have recorded the constraints between the intervals.

**EnforcePathConsistency1** $(\mathcal{C})$:
*Input:* a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
*Output:* an equivalent, but path consistent CSP $\mathcal{C}'$

**repeat**
  **for** each pair $(i,j)$, $1 \leq i,j \leq n$
    **for** each $k$ with $1 \leq k \leq n$
      *Table* $[i,j]$ := *Table* $[i,j] \cap$ ( *Table* $[i,k] \circ$ *Table* $[k,j]$ )
    **endfor**
  **endfor**
**until** no entry in *Table* is changed

- ⇝ terminates;
- ⇝ needs $O(n^5)$ intersections and compositions.





# An $O(n^3)$ Algorithm

**EnforcePathConsistency2** $(\mathcal{C})$:
*Input:* a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$
*Output:* an equivalent, but path consistent CSP $\mathcal{C}'$

*Paths* $(i,j) = \{(i,j,k) \,:\, 1 \leq k \leq n\} \cup \{(k,i,j) \,:\, 1 \leq k \leq n\}$

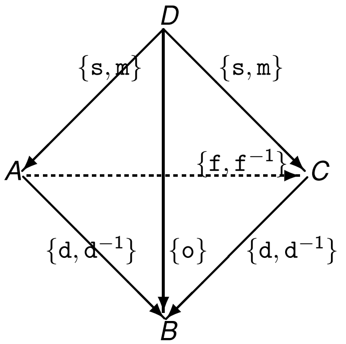
*Queue* := $\bigcup_{i,j}$ *Paths* $(i,j)$

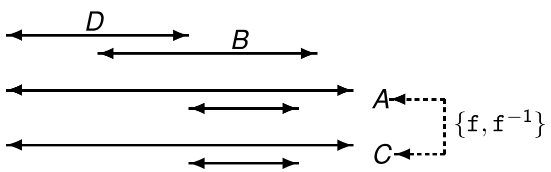
**While** $Q \neq \emptyset$
  select and delete $(i,k,j)$ from $Q$
  $T$ := *Table* $[i,j] \cap$ ( *Table* $[i,k] \circ$ *Table* $[k,j]$ )
  **if** $T \neq$ *Table* $[i,j]$
    *Table* $[i,j] := T$
    *Table* $[j,i] := T^{-1}$
    *Queue* := *Queue* $\cup$ *Paths* $(i,j)$
  **endif**
**endwhile**





# Example for Incompleteness








# NP-Hardness

## Theorem (Kautz & Vilain)

*CSAT is NP-hard for Allen's interval calculus.*

## Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability.
Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

| | | | |
|---|---|---|---|
| 1 | $\{\mathtt{m}\}$ | 2 | |
| 2 | $\{\mathtt{m}\}$ | 3 | |
| $v_i$ | $\{\mathtt{m}, \equiv, \mathtt{m}^{-1}\}$ | 2 | $\forall v_i \in V$ |
| $v_i$ | $\{\mathtt{m}, \mathtt{m}^{-1}, \prec, \succ\}$ | $v_j$ | $\forall (v_i, v_j) \in E$ |

This constraint system is satisfiable *iff* $G$ can be colored with 3 colors. □

# Looking for Special Cases

- *Idea*: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- Note: Interval formulae $X\ R\ Y$ can be expressed as clauses over atoms of the form $a\ op\ b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $op \in \{<, >, =, \leq, \geq\}$.
- Example: All base relations can be expressed as unit clauses.

**Lemma**
*Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.*



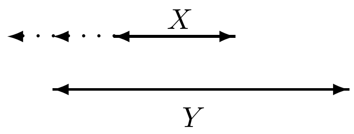


# The Continuous Endpoint Class

Continuous Endpoint Class $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}$, because

$$\pi(X\ \{\mathtt{d}, \mathtt{o}, \mathtt{s}\}\ Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^+\}$$








# Why Do We Have Completeness?

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a *sub-algebra* wrt. these three operations on relations). This can be shown by using a computer program.

**Lemma**
*Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.*

**Theorem (van Beek)**
*Path consistency solves CMIN($\mathcal{C}$) and decides CSAT($\mathcal{C}$).*

**Proof.**
Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal. □

**Corollary**
*A path consistent interval CSP consisting of base relations only is satisfiable.*





# Helly's Theorem

**Definition**
A set $M \subseteq \mathbf{R}^n$ is *convex* iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

**Theorem (Helly)**
*Let $F$ be a family of at least $n + 1$ convex sets in $\mathbf{R}^n$. If all sub-families of $F$ with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.*

# Strong $n$-Consistency (1)

Proof.

We prove the claim by induction over $k$ with $k \leq n$.

*Base case:* $k = 1, 2, 3$ $\surd$

*Induction assumption:* Assume strong $k-1$-consistency (and non-emptiness of all relations)

*Induction step:* From the assumption, it follows that there is an instantiation of $k-1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k-1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.





# Strong $n$-Consistency (2): Instantiating the $k$th Variable

Proof (Part 2).

The instantiation of the $k-1$ variables $X_i$ to $(s_i, e_i)$ restricts the instantiation of $X_k$.

*Note:* Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \ \wedge \ e_j \geq X_k^- \ \wedge \ \ldots$$

Such inequalities define convex subsets in $\mathbf{R}^2$.

⇝ Consider sets of 3 inequalities ($=$ 3 convex sets).





# Strong $n$-Consistency (3): Using Helly's Theorem

Proof (Part 3).

*Case 1:* All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

*Case 2:* The inequalities mention $X_k^-$ and $X_k^+$, but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

*Case 3*: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by the same argument as above there exists a common point.

⇝ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.

⇝ Strong $k$-consistency for all $k \leq n$. □





# Outlook

- CMIN($\mathcal{C}$) can be computed in $O(n^3)$ time (for $n$ being the number of intervals) using the path consistency algorithm.
- $\mathcal{C}$ is a set of relations occurring "naturally" when observations are uncertain.
- $\mathcal{C}$ contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes

# A Maximal Tractable Sub-Algebra





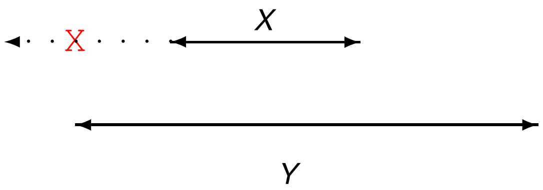


# The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\begin{aligned}\pi(X\ \{\mathtt{d},\mathtt{o}\}\ Y) \ = \ & \{X^- < X^+, Y^- < Y^+, \\ & X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ & X^+ < Y^+\}\end{aligned}$$




**Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)**
*The path-consistency method decides CSAT($\mathcal{P}$).*





# The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only Horn clauses , where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}$:

$$\pi(X\{\mathtt{o},\mathtt{s},\mathtt{f}^{-1}\}Y) = \begin{cases} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{cases}.$$





# Partial Orders: The $ORD$ Theory

Let $ORD$ be the following theory:

$$\begin{array}{llll} \forall x,y,z: & x \leq y \wedge y \leq z & \rightarrow x \leq z & \textit{(transitivity)} \\ \forall x: & x \leq x & & \textit{(reflexivity)} \\ \forall x,y: & x \leq y \wedge y \leq x & \rightarrow x = y & \textit{(anti-symmetry)} \\ \forall x,y: & x = y & \rightarrow x \leq y & \textit{(weakening of =)} \\ \forall x,y: & x = y & \rightarrow y \leq x & \textit{(weakening of =).} \end{array}$$

- $ORD$ describes partially ordered sets, $\leq$ being the ordering relation.
- $ORD$ is a Horn theory
- What is missing wrt to *dense* and *linear* orders?

# Satisfiability over Partial Orders

## Proposition

*Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.*

## Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy $ORD$), this direction is trivial.
$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □





# Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

## Proposition

$ORD \cup \pi(\Theta)$ *is satisfiable iff* $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ *is so.*

## Proof idea: Herbrand expansion!

□

## Theorem

*CSAT($\mathcal{H}$) can be decided in polynomial time.*

## Proof.

CSAT($\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. □

$$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H} \quad \text{with} \quad |\mathcal{C}| = 83,\ |\mathcal{P}| = 188,\ |\mathcal{H}| = 868$$





# Path-Consistency and the OH-Class

## Lemma

*Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then*

$$(X\{\}Y) \notin \Theta \quad \text{iff} \quad \Theta \text{ is satisfiable}$$

## Proof Idea.

One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows. □

## Lemma

*$\mathcal{H}$ is closed under intersection, composition, and conversion.*

## Theorem

*The path-consistency method decides CSAT($\mathcal{H}$).*

- ⇝ Maximality of $\mathcal{H}$?
- ⇝ Do we have to check all 8192 - 868 extensions?





# Complexity of Sub-Algebras

Let $\hat{S}$ be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by $S$)

## Theorem

*CSAT($\hat{S}$) can be polynomially transformed to CSAT($S$).*

## Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in $S$, introducing perhaps some fresh variables. □

- ⇝ Polynomiality of $S$ extends to $\hat{S}$.
- ⇝ NP-hardness of $\hat{S}$ is inherited by all generating sets $S$.
- ⇝ Note: $\mathcal{H} = \hat{\mathcal{H}}$.

## Minimal Extensions of the $\mathcal{H}$-Subclass

A *computer-aided* case analysis leads to the following result:

Lemma
*There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$*

$$N_1 = \{\mathtt{d}, \mathtt{d}^{-1}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}\} \in \mathcal{X}_1$$
$$N_2 = \{\mathtt{d}^{-1}, \mathtt{o}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

Theorem
*CSAT($\mathcal{H} \cup \{N_i\}$) is NP-complete.*

Question: Are there other maximal tractable subclasses?





## “Interesting” Subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations.

A *computer-aided* case analysis reveals: For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. $N_1$ or $N_2$ is in $\hat{S}$.

In case 2, one can show: CSAT($S$) is NP-complete.

⇝ $\mathcal{H}$ is the only maximal tractable subclass that is interesting.

Meanwhile, there is a complete classification of all sub-algebras containing at least one basic relation [IJCAI 2001] . . . but the question for sub-algebras not containing a basic relation is open.





## Relevance?

Theoretical:

We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*. ⊕

Practical: All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$! ⊖

Backtracking methods might profit from the result because the branching factor is lower. ?

⇝ How difficult is CSAT($\mathcal{A}$) in practice?

⇝ What are the relevant branching factors?





## Solving General Allen CSPs

- Backtracking algorithm using path-consistency as a *forward-checking method*
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

⇝ Which tractable fragment should one use?

# Branching Factors

- If the labels are split into base relations, then on average a label is split into

  **6.5 relations**
- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into

  **2.955 relations**
- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into

  **2.533 relations**

$\rightsquigarrow$ A difference of 0.422

$\rightsquigarrow$ This makes a difference for "hard" instances.





# Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.





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