# Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models).
- *Nonmonotonicity* is only a negative characterization: If we have $\Theta \mathrel{|\!\sim} \varphi$, we do not necessarily have $\Theta \cup \{\psi\} \mathrel{|\!\sim} \varphi$.
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- Could we have a constructive positive characterization of default reasoning?

# Plausible Consequences

- In conventional logic, we have the logical consequence relation $\alpha \models \beta$: If $\alpha$ is true, then also $\beta$ is true.
- Instead, we will study the relation of plausible consequence $\alpha \mid\sim \beta$: if $\alpha$ is all we know, can we conclude $\beta$?
- $\alpha \mid\sim \beta$ does not imply $\alpha \wedge \alpha' \mid\sim \beta$!!!
  Compare to conditional probability:
  $P(\beta|\alpha) \neq P(\beta|\alpha, \alpha')$!
- Find rules characterizing $\mid\sim$: for example, if $\alpha \mid\sim \beta$ and $\alpha \mid\sim \gamma$, then $\alpha \mid\sim \beta \wedge \gamma$.
- Write down all such rules!
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# Desirable Properties 1: Reflexivity



- Reflexivity:

$$\frac{}{\alpha \mid\sim \alpha}$$

  - Rationale: If $\alpha$ holds, this *normally implies* $\alpha$.
  - Example: `Tom goes to a party` *normally implies* that `Tom goes to a party`.

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# Reflexivity in Default Logic

## Plausible consequence as Reasoning in Default Logic

Let us consider relations $\mid\!\sim_\Delta$ that are defined in terms of Default Logic.

$\alpha \mid\!\sim_{\langle D,W\rangle} \beta$ means that $\beta$ is a skeptical conclusion of $\langle D, W \cup \{\alpha\}\rangle \mid\!\sim \beta$.

## Proposition

*Default Logic satisfies Reflexivity.*

## Proof.

The question is: does $\alpha$ skeptically follow from $\Delta = \langle D, W \cup \{\alpha\}\rangle$?

For all extensions $E$ of $\Delta$, $W \cup \{\alpha\} \subseteq E$ by definition. Hence $\alpha \in E$ and $\alpha$ belongs to all extensions of $\Delta$. □

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# Desirable Properties 2: Left Logical Equivalence

- Left Logical Equivalence:

$$\frac{\models \alpha \leftrightarrow \beta,\ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$$

  - Rationale: It is not the syntactic form, but the logical content that is responsible for what we conclude normally.
  - Example: Assume that `Tom goes` *or* `Peter goes` *normally implies* `Mary goes`.
    Then we would expect that `Peter goes` *or* `Tom goes` *normally implies* `Mary goes`.

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Assume that $\models \alpha \leftrightarrow \beta$ and $\gamma$ is in all extensions of $\langle D, W \cup \{\alpha\}\rangle$. The definition of extensions is invariant under replacing any formula by an equivalent formula. Hence $\langle D, W \cup \{\beta\}\rangle$ has exactly the same extensions, and $\gamma$ is in every one of them. □

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# Desirable Properties 3: Right Weakening

- Right Weakening:

$$\frac{\models \alpha \rightarrow \beta, \; \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}$$

  - Rationale: If something can be concluded normally, then everything classically implied should also be concluded normally.
  - Example: Assume that `Mary goes` *normally implies* `Clive goes` *and* `John goes`.
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  - From 1 & 3 supraclassicality follows:

$$\alpha \mid\!\sim \alpha \quad + \quad \frac{\models \alpha \rightarrow \beta, \; \alpha \mid\!\sim \alpha}{\alpha \mid\!\sim \beta} \quad \Rightarrow \quad \frac{\alpha \models \beta}{\alpha \mid\!\sim \beta}$$

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# Right Weakening in Default Logic





**Proposition**

*Default Logic satisfies Right Weakening.*

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Assume $\alpha$ is in all extensions of a default theory $\langle D, W \cup \{\gamma\}\rangle$ and $\models \alpha \rightarrow \beta$. Extensions are closed under logical consequence. Hence also $\beta$ is in all extensions. □

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- Cut:

$$\frac{\alpha \mid\!\sim \beta, \ \alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

  - Rationale: If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.
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**Proof idea.**

Show that every extension $E$ of $\Delta = \langle D, W \cup \{\alpha\}\rangle$ is also an extension of $\Delta' = \langle D, W \cup \{\alpha \wedge \beta\}\rangle$.

*The consistency of the justifications of defaults are tested against $E$ both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \wedge \beta\}$ case.*

*The preconditions that are derivable when starting from $W \cup \{\alpha\}$ are also derivable when starting from $W \cup \{\alpha \wedge \beta\}$. $W \cup \{\alpha \wedge \beta\}$ does not allow deriving further preconditions because also with $W \cup \{\alpha\}$ at some point $\beta$ is derived.* Hence $E$ is also an extension of $\Delta'$.

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# Desirable Properties 5: Cautious Monotonicity

- Cautious Monotonicity:

$$\frac{\alpha \mid\!\sim \beta, \; \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$

  - Rationale: In general, adding new premises may cancel some conclusions. However, existing conclusions may be added to the premises without canceling any conclusions!
  - Example: Assume that `Mary goes` *normally implies* `Clive goes` and `Mary goes` *normally implies* `John goes`. `Mary goes` *and* `Jack goes` might not *normally imply* that `John goes`. However, `Mary goes` and `Clive goes` should *normally imply* that `John goes`.

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# Cautious Monotonicity in Default Logic





## Proposition

*Default Logic* **does not** *satisfy Cautious Monotonicity.*

## Proof.

Consider the default theory $\langle D, W\rangle$ with

$$D = \left\{ \frac{a : g}{g}, \frac{g : b}{b}, \frac{b : \neg g}{\neg g} \right\} \text{ and } W = \{a\}.$$

$E = \mathrm{Th}(\{a, b, g\})$ is the only extension of $\langle D, W\rangle$ and $g$ follows skeptically.
For $\langle D, W \cup \{b\}\rangle$ also $\mathrm{Th}(\{a, b, \neg g\})$ is an extension, and $g$ does not follow skeptically. □

# Cumulativity





## Lemma

*Rules 4 & 5 can be equivalently stated as follows.*

> *If $\alpha \mathrel{|\!\sim} \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \wedge \beta$ are identical.*

The above property is also called cumulativity.

## Proof.

$\Rightarrow$: Assume that 4 & 5 hold and $\alpha \mathrel{|\!\sim} \beta$. Assume further that $\alpha \mathrel{|\!\sim} \gamma$. With rule 5 (CM), we have $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$. Similarly, from $\alpha \wedge \beta \mathrel{|\!\sim} \gamma$ by rule 4 (Cut) we get $\alpha \mathrel{|\!\sim} \gamma$.
Hence the plausible conclusions from $\alpha$ and $\alpha \wedge \beta$ are the same.
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**Proof.**

$\Rightarrow$: Assume that 4 & 5 hold and $\alpha \mid\!\sim \beta$. Assume further that $\alpha \mid\!\sim \gamma$. With rule 5 (CM), we have $\alpha \wedge \beta \mid\!\sim \gamma$. Similarly, from $\alpha \wedge \beta \mid\!\sim \gamma$ by rule 4 (Cut) we get $\alpha \mid\!\sim \gamma$.
Hence the plausible conclusions from $\alpha$ and $\alpha \wedge \beta$ are the same.
$\Leftarrow$. Assume Cumulativity and $\alpha \mid\!\sim \beta$. Now we can derive *rules 4 and 5*. □

# Cumulativity





**Lemma**

*Rules 4 & 5 can be equivalently stated as follows.*

> *If $\alpha \mid\!\sim \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \wedge \beta$ are identical.*

The above property is also called cumulativity.

**Proof.**

$\Rightarrow$: Assume that 4 & 5 hold and $\alpha \mid\!\sim \beta$. Assume further that $\alpha \mid\!\sim \gamma$. With rule 5 (CM), we have $\alpha \wedge \beta \mid\!\sim \gamma$. Similarly, from $\alpha \wedge \beta \mid\!\sim \gamma$ by rule 4 (Cut) we get $\alpha \mid\!\sim \gamma$.
Hence the plausible conclusions from $\alpha$ and $\alpha \wedge \beta$ are the same.
$\Leftarrow$. Assume Cumulativity and $\alpha \mid\!\sim \beta$. Now we can derive *rules 4 and 5*. □

# The System C

1. Reflexivity

$$\frac{}{\alpha \mid\!\sim \alpha}$$

2. Left Logical Equivalence

$$\frac{\models \alpha \leftrightarrow \beta, \ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$$

3. Right Weakening

$$\frac{\models \alpha \to \beta, \ \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}$$

4. Cut

$$\frac{\alpha \mid\!\sim \beta, \ \alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

5. Cautious Monotonicity

$$\frac{\alpha \mid\!\sim \beta, \ \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$

# Derived Rules in **C**

- Equivalence:

$$\frac{\alpha \mid\!\sim \beta,\ \beta \mid\!\sim \alpha,\ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$$

- And:

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$$

- MPC:

$$\frac{\alpha \mid\!\sim \beta \rightarrow \gamma,\ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$

# Derived Rules in C

- Equivalence:

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- And:

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# Proofs

## Equivalence

| | |
|---|---|
| Assumption: | $\alpha \mid\!\sim \beta, \quad \beta \mid\!\sim \alpha, \quad \alpha \mid\!\sim \gamma$ |
| Cautious Monotonicity: | $\alpha \wedge \beta \mid\!\sim \gamma$ |
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| Cut: | $\alpha \wedge \beta \mid\!\sim \beta \wedge \gamma$ |
| Cut: | $\alpha \mid\!\sim \beta \wedge \gamma$ |

MPC is an Exercise.

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# Undesirable Properties 1: Monotonicity and Contraposition

- Monotonicity:

$$\frac{\models \alpha \rightarrow \beta,\ \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

  - Example: Let us assume that
    `John goes` *normally implies* `Mary goes`.
    Now we will probably not expect that
    `John goes` *and* `Joan` (who is not in talking terms with Mary) `goes` *normally implies* `Mary goes`.

- Contraposition:

$$\frac{\alpha \mid\!\sim \beta}{\neg\beta \mid\!\sim \neg\alpha}$$

  - Example: Let us assume that
    `John goes` *normally implies* `Mary goes`.
    Would we expect that
    `Mary does not go` *normally implies* `John does not go`?
    What if `John goes` always?

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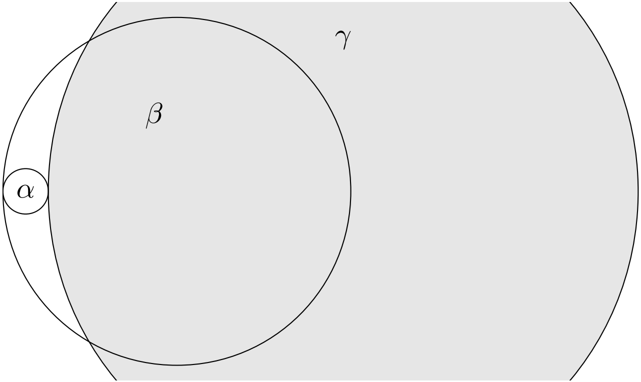
# Undesirable Properties 1: Monotonicity

$\alpha \models \beta$, $\beta \mid\!\sim \gamma$ but not $\alpha \mid\!\sim \gamma$ pictorially:



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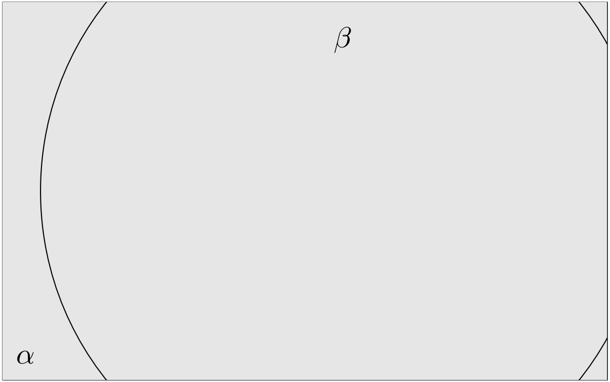

# Undesirable Properties 2: Transitivity & EHD

- Transitivity:

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  - Example: Let us assume that `John goes` *normally implies* `Mary goes` and `Mary goes` *normally implies* `Jack goes`. Now, should `John goes` *normally imply* that `Jack goes`? If `John goes` very seldom?

- Easy Half of Deduction Theorem (*EHD*):

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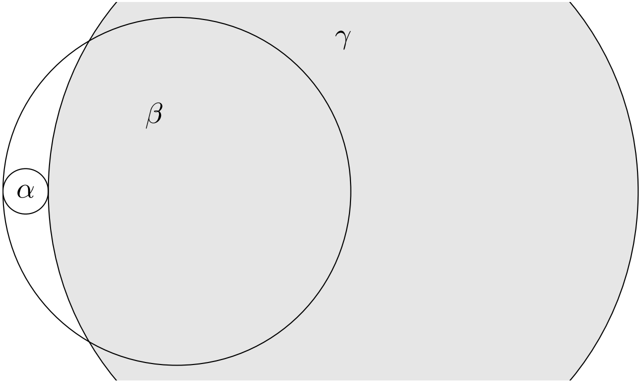

# Undesirable Properties 2: EHD

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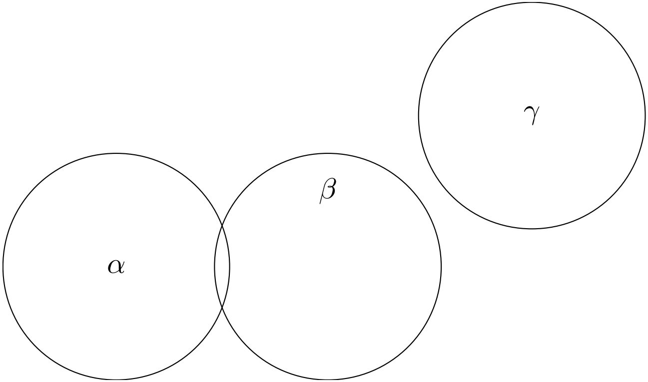

# Undesirable Properties 3

**Theorem**

*In the presence of the rules in system C, monotonicity and EHD are equivalent.*

**Proof.**

*Monotonicity ⇒ EHD*:

- $\alpha \mid\sim \beta \rightarrow \gamma$ (assumption)
- $\alpha \wedge \beta \mid\sim \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \wedge \beta \mid\sim \alpha \wedge \beta$ (reflexivity)
- $\alpha \wedge \beta \mid\sim \beta$ (right weakening)
- $\alpha \wedge \beta \mid\sim \gamma$ (MPC)

*Monotonicity ⇐ EHD*:

- $\models \alpha \rightarrow \beta, \beta \mid\sim \gamma$ (assumption)
- $\beta \mid\sim \alpha \rightarrow \gamma$ (right weakening)
- $\beta \wedge \alpha \mid\sim \gamma$ (EHD)
- $\alpha \mid\sim \gamma$ (left logical equivalence)

□

## Undesirable Properties 3





### Theorem

*In the presence of the rules in system C, monotonicity and EHD are equivalent.*

### Proof.

*Monotonicity* $\Rightarrow$ *EHD*:

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- $\alpha \wedge \beta \mid\sim \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \wedge \beta \mid\sim \alpha \wedge \beta$ (reflexivity)
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*In the presence of* right weakening, contraposition *implies* monotonicity*.*

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Note: *Monotonicity* does not imply *contraposition*, even in the presence of all rules of system C!

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# Cumulative Closure 1

- How do we *reason* with $\mid\!\sim$ from $\varphi$ to $\psi$?
- Assumption: We have a set K of conditional statements of the form $\alpha \mid\!\sim \beta$.
  The question is: Assuming the statements in $\mathbf{K}$, is it plausible to conclude $\psi$ given $\varphi$?
- Idea: We consider all cumulative consequence relations that contain $\mathbf{K}$.
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**Lemma**

*The set of cumulative consequence relations is closed under intersection.*

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Let $\mid\!\sim_1$ and $\mid\!\sim_2$ be cumulative consequence relations. We have to show that $\mid\!\sim_1 \cap \mid\!\sim_2$ is a cumulative consequence relation, that is, it satisfies the rules 1-5.

Take any instance of the any of the rules. If the preconditions are satisfied by $\mid\!\sim_1$ and $\mid\!\sim_2$, then the consequence is trivially also satisfied by both. □

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**Theorem**

*For each finite set of conditional statements $\mathbf{K}$, there exists a unique smallest cumulative consequence relation containing $\mathbf{K}$.*

**Proof.**

Assume the contrary, i.e., there are incomparable minimal sets $\mathbf{K}_1, \ldots, \mathbf{K}_m$. Then $\mathbf{K} = \mathbf{K}_1 \cap \ldots \cap \mathbf{K}_m$ is a unique smallest cumulative consequence relation containing $\mathbf{K}$: contradiction. This relation is the cumulative closure $\mathbf{K}^C$ of $\mathbf{K}$. □

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# Cumulative Models – informally

- We will now try to characterize cumulative reasoning model-theoretically.
- Idea: Cumulative models consist of states ordered by a preference relation.
- *States* characterize beliefs.
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- We say: $\alpha \mid\!\sim \beta$ is accepted in a model if in all most preferred states in which $\alpha$ is true, also $\beta$ is true.

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# Preference Relation

- Let $\prec$ be a binary relation on a set $U$.
  $\prec$ is asymmetric iff

$$s \prec t \text{ implies } t \not\prec s \quad \text{for all } s, t \in U.$$

- Let $V \subseteq U$ and $\prec$ be a binary relation on $U$.
  - $t \in V$ is minimal in $V$ iff $s \not\prec t$ for all $s \in V$.
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- Let $P \subseteq U$ and $\prec$ be a binary relation on $U$.
  $P$ is smooth iff for all $t \in P$, either $t$ is minimal in $P$ or there is $s \in P$ such that $s$ is minimal in $P$ and $s \prec t$.
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  $P$ is smooth iff for all $t \in P$, either $t$ is minimal in $P$ or there is $s \in P$ such that $s$ is minimal in $P$ and $s \prec t$.
- Note: $\prec$ is not a partial order but an arbitrary relation!

# Preference Relation

- Let $\prec$ be a binary relation on a set $U$.
  $\prec$ is asymmetric iff

$$s \prec t \text{ implies } t \not\prec s \quad \text{for all } s, t \in U.$$

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# Cumulative Models – formally

- Let $\mathcal{U}$ be the set of all *possible worlds (propositional interpretations)*.
- A cumulative model $W$ is a triple $\langle S, l, \prec \rangle$ such that
  1. $S$ is a set of states,
  2. $l$ is a mapping $l : S \to 2^{\mathcal{U}}$,and
  3. $\prec$ is an arbitrary *binary relation*

  such that the smoothness condition is satisfied (see below).
- A state $s \in S$ satisfies a formula $\alpha$ ($s \models \alpha$) iff $m \models \alpha$ for all propositional interpretations $m \in l(s)$.
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# Consequence Relation Induced by a Cumulative Model

A cumulative model $W$ *induces a consequence* relation $\mid\!\sim_W$ as follows:

$$\alpha \mid\!\sim_W \beta \text{ iff } s \models\!\!\!\equiv \beta \text{ for every minimal } s \text{ in } \widehat{\alpha}.$$

**Example**

Model $W = \langle\{s_1, s_2, s_3\}, l, \prec\rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

$$\begin{aligned} l(s_1) &= \{\{\neg p, b, f\}\} \\ l(s_2) &= \{\{p, b, \neg f\}\} \\ l(s_3) &= \{\{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}\} \end{aligned}$$

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$\neg p \wedge \neg b \mid\!\sim f$? N Also: $\neg p \wedge \neg b \not\mid\!\sim \neg f$!

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**Theorem**

*If $W$ is a cumulative model, then $\mid\!\sim_W$ is a cumulative consequence relation.*

**Proof.**

- *Reflexivity:* satisfied $\surd$.
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- *Cut:* $\alpha \mid\!\sim \beta, \alpha \wedge \beta \mid\!\sim \gamma \Rightarrow \alpha \mid\!\sim \gamma$. Assume that all minimal elements of $\widehat{\alpha}$ satisfy $\beta$, and all minimal elements of $\widehat{\alpha \wedge \beta}$ satisfy $\gamma$. Every minimal element of $\widehat{\alpha}$ satisfies $\alpha \wedge \beta$.Since $\widehat{\alpha \wedge \beta} \subseteq \widehat{\alpha}$, all minimal elements of $\widehat{\alpha}$ are also minimal elements of $\widehat{\alpha \wedge \beta}$. Hence $\alpha \mid\!\sim_W \gamma$.

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## Proof continues...

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  Assume $\alpha \mid\sim_W \beta$ and $\alpha \mid\sim_W \gamma$. We have to show: $\alpha \wedge \beta \mid\sim_W \gamma$, i.e., $s \models \gamma$ for all minimal $s \in \widehat{\alpha \wedge \beta}$.

  Clearly, every minimal $s \in \widehat{\alpha \wedge \beta}$ is in $\widehat{\alpha}$.

  We show that every minimal $s \in \widehat{\alpha \wedge \beta}$ is *minimal* in $\widehat{\alpha}$.

  *Assumption:* There is $s$ that is minimal in $\widehat{\alpha \wedge \beta}$ but not minimal in $\widehat{\alpha}$. Because of *smoothness* there is minimal $s' \in \widehat{\alpha}$ such that $s' \prec s$. We know, however, that $s' \models \beta$, which means that $s' \in \widehat{\alpha \wedge \beta}$. Hence $s$ is not minimal in $\widehat{\alpha \wedge \beta}$. Contradiction! Hence $s$ must be minimal in $\widehat{\alpha}$, and therefore $s \models \gamma$. Because this is true for all minimal elements in $\widehat{\alpha \wedge \beta}$, we get $\alpha \wedge \beta \mid\sim_W \gamma$.

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# Consequence: Counterexamples

Now we have a method for showing that a principle does not hold for cumulative consequence relations. $\rightsquigarrow$ Simply construct a cumulative model that falsifies the principle.

Contraposition: $\alpha \mid\!\sim \beta \Rightarrow \neg\beta \mid\!\sim \neg\alpha$

$$\begin{aligned}
W &= \langle S, l, \prec \rangle \\
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- Each cumulative model $W$ *induces* a cumulative consequence relation $\mid\!\sim_W$.
- Problem: Can we generate all cumulative consequence relations in this way?
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# Transitivity of the Preference Relation?

- Could we strengthen the preference relation to transitive relations without sacrificing anything?
  No!
- In such models, the following additional principle called Loop is valid:

$$\frac{\alpha_o \mid\sim \alpha_1, \alpha_1 \mid\sim \alpha_2, \ldots, \alpha_k \mid\sim \alpha_0}{\alpha_0 \mid\sim \alpha_k}$$

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Or rule:

$$\frac{\alpha \mathrel{|\!\sim} \gamma,\ \beta \mathrel{|\!\sim} \gamma}{\alpha \vee \beta \mathrel{|\!\sim} \gamma}$$

Not true in C. Counterexample:

$$\begin{aligned} W &= \langle S, l, \prec \rangle \\ S &= \{s_1, s_2, s_3\}, s_i \not\prec s_j \ \forall s_i, s_j \in S \\ l(s_1) &= \{\{a, b, c\}, \{a, \neg b, c\}\} \\ l(s_2) &= \{\{b, a, c\}, \{b, \neg a, c\}\} \\ l(s_3) &= \{\{a, b, \neg c\}, \{a, \neg b, \neg c\}, \{\neg a, b, \neg c\}\} \end{aligned}$$

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# System P

- System P contains all rules of C and the Or rule.
- Derived rules in P:
  - Hard half of deduction theorem (S):

$$\frac{\alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \rightarrow \gamma}$$

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# Preferential Models



**Definition**

A cumulative model $W = \langle S, l, \prec \rangle$ such that $\prec$ is a *strict partial order* (irreflexive and transitive) and $|l(s)| = 1$ for all $s \in S$ is a preferential model.

# Preferential Models



**Theorem (Soundness)**

*The consequence relation $\mid\!\sim_W$ induced by a preferential model is preferential.*

**Proof.**

Since $W$ is cumulative, we only have to verify that Or holds. Note that in preferential models we have $\widehat{\alpha \vee \beta} = \widehat{\alpha} \cup \widehat{\beta}$. Suppose $\alpha \mid\!\sim_W \gamma$ and $\beta \mid\!\sim_W \gamma$. Because of the above equation, each minimal state of $\widehat{\alpha \vee \beta}$ is minimal in $\widehat{\alpha} \cup \widehat{\beta}$. Since $\gamma$ is satisfied in all minimal states in $\widehat{\alpha} \cup \widehat{\beta}$, $\gamma$ is also satisfied in all minimal states of $\widehat{\alpha \vee \beta}$. Hence $\alpha \vee \beta \mid\!\sim_W \gamma$. □

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Since $W$ is cumulative, we only have to verify that Or holds. Note that in preferential models we have $\widehat{\alpha \vee \beta} = \widehat{\alpha} \cup \widehat{\beta}$. Suppose $\alpha \mid\!\sim_W \gamma$ and $\beta \mid\!\sim_W \gamma$. Because of the above equation, each minimal state of $\widehat{\alpha \vee \beta}$ is minimal in $\widehat{\alpha} \cup \widehat{\beta}$. Since $\gamma$ is satisfied in all minimal states in $\widehat{\alpha} \cup \widehat{\beta}$, $\gamma$ is also satisfied in all minimal states of $\widehat{\alpha \vee \beta}$. Hence $\alpha \vee \beta \mid\!\sim_W \gamma$. □

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# Summary of Consequence Relations

| *System* | *Models* |
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| Reflexivity | States: sets of worlds |
| Left Logical Equivalence | Preference relation: arbitrary |
| Right Weakening | Models must be smooth |
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# Strengthening the Consequence Relation

- System C and System P do not produce many of the inferences one would hope for:

  *Given* $K = \{$*Bird* $\mid\!\sim$ *Flies*$\}$ *one cannot conclude Red* $\wedge$ *Bird* $\mid\!\sim$ *Flies!!*

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- The rules so far seem to be reasonable and one cannot think of rules of the same form (if we have some plausible implications, other plausible implications should hold) that could be added.
- However, there are other types of rules one might want add.
- Disjunctive Rationality:

$$\frac{\alpha \not\mid\!\sim \gamma\ ,\ \beta \not\mid\!\sim \gamma}{\alpha \vee \beta \not\mid\!\sim \gamma}$$

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# Probabilistic View of Plausible Consequences

- Consider probability distributions $P$ on the set $\mathcal{M}$ of all interpretations $m \in \mathcal{M}$ (worlds) of our language.
- $P(m)$ is the probability of the possible world $m$.
- Extend this to probability of formulae:

$$P(\alpha) = \sum \{P(m) | m \in \mathcal{M}, m \models \alpha\}$$

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# $\epsilon$-Entailment



**Definition**

$\alpha \mid\!\sim \beta$ is $\epsilon$-entailed by a set $K$ iff for all $\epsilon > 0$ there is $\delta > 0$ such that $P(\beta|\alpha) \geq 1 - \epsilon$ for all probability distributions $P$ such that $P(\beta'|\alpha') \geq 1 - \delta$ for all $\alpha' \mid\!\sim \beta' \in K$.

# $\epsilon$-Entailment: Example

One probability distribution $P$ such that $P(f|b) \geq 0.9$, $P(\neg f|p) \geq 0.9$ and $P(b|p) \geq 0.9$ is the following.

| | $p$ | $b$ | $f$ | $P$ |
|---|---|---|---|---|
| $w_1$ | 0 | 0 | 0 | 0.00 |
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| $w_4$ | 0 | 1 | 1 | 0.99 |
| $w_5$ | 1 | 0 | 0 | 0.00 |
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# Properties of $\epsilon$-Entailment

**Theorem**

*$\alpha \mid\!\sim \beta$ is in all preferential consequence relations that include $K$ if and only if $\alpha \mid\!\sim \beta$ is $\epsilon$-entailed by $K$.*

So, System P provides a proof system that exactly corresponds to $\epsilon$-entailment.

# Weakness of $\epsilon$-Entailment

- Question: Why is Eagle $\mid\!\sim$ Flies not $\epsilon$-entailed by $K = \{\text{Eagle} \mid\!\sim \text{Bird}, \text{Bird} \mid\!\sim \text{Flies}\}$?
- Answer: Because there are probability distributions that simultaneously assign very high probabilities to $P(\text{Bird}|\text{Eagle})$ and $P(\text{Flies}|\text{Bird})$ and a low probability to $P(\text{Flies}|\text{Eagle})$.
- $K$ does not justify the low probability of $P(\text{Flies}|\text{Eagle})$: there are exactly as many worlds satisfying $\text{Bird} \wedge \text{Eagle} \wedge \text{Flies}$ and $\text{Bird} \wedge \text{Eagle} \wedge \neg\text{Flies}$, and the worlds satisfying $\text{Bird} \wedge \text{Flies}$ have a much higher probability than those satisfying $\text{Bird} \wedge \neg\text{Flies}$. Why should the probabilities for eagles be the other way round?
- We would like to restrict to probability distributions that are not biased toward non-flying eagles without a reason.

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# Entropy of a Probability Distribution



**Definition**

The entropy of a probability distribution $P$ is

$$H(P) = -\sum_{m \in \mathcal{M}} P(m) \log P(m)$$

The probability distribution with the highest entropy is the one that assigns the same probability to every world.

# ME-Entailment



**Definition**

$\alpha \mid\!\sim \beta$ is ME-entailed by a set $K$ iff for all $\epsilon > 0$ there is $\delta > 0$ such that $P(\beta|\alpha) \geq 1 - \epsilon$ for the distribution $P$ that has the maximum entropy among distributions satisfying $P(\beta'|\alpha') \geq 1 - \delta$ for all $\alpha' \mid\!\sim \beta' \in K$.

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The distribution $P$ that has the maximum entropy among distributions such that $P(b|e) \geq 0.9$ and $P(f|b) \geq 0.9$ is the following.

| | $e$ | $b$ | $f$ | $P$ |
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# ME-Entailment: Examples



1. $\{\text{Eagle} \mid\!\sim \text{Bird}, \text{Bird} \mid\!\sim \text{Flies}\}$ ME-entails $\text{Eagle} \mid\!\sim \text{Flies}$
2. $\{\text{Penguin} \mid\!\sim \text{Bird}, \text{Bird} \mid\!\sim \text{Flies}, \text{Penguin} \mid\!\sim \neg\text{Flies}\}$ ME-entails $\text{Bird} \wedge \text{Penguin} \mid\!\sim \neg\text{Flies}$
3. $\{\text{Eagle} \mid\!\sim \text{Bird}\}$ ME-entails $\neg\text{Bird} \mid\!\sim \neg\text{Eagle}$
4. $\{\text{Bat} \mid\!\sim \text{Mammal}, \text{Bat} \mid\!\sim \text{Winged-Animal}, \text{Winged-Animal} \mid\!\sim \text{Flies}, \text{Mammal} \mid\!\sim \neg\text{Flies}\}$ does not ME-entail $\text{Bat} \mid\!\sim \text{Flies}$.

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2. $\{Penguin \mid\!\sim Bird, Bird \mid\!\sim Flies, Penguin \mid\!\sim \neg Flies\}$ ME-entails $Bird \wedge Penguin \mid\!\sim \neg Flies$
3. $\{Eagle \mid\!\sim Bird\}$ ME-entails $\neg Bird \mid\!\sim \neg Eagle$
4. $\{Bat \mid\!\sim Mammal, Bat \mid\!\sim Winged\text{-}Animal, Winged\text{-}Animal \mid\!\sim Flies, Mammal \mid\!\sim \neg Flies\}$ does not ME-entail $Bat \mid\!\sim Flies$.

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# ME-Entailment: Problems

- No good proof systems and reasoning algorithms exist! Computing maximum entropy distributions is computationally extremely complex when the number of worlds is high.
- The way knowledge is expressed still has a strong impact on what conclusions can be derived.

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does not ME-entail anything about flying ability of bats.

$$\{ \text{Bat} \mid\!\sim \text{Mammal}, \quad \text{Mammal} \mid\!\sim \neg\text{Winged-Animal}, \quad \text{Bat} \mid\!\sim \text{Winged-Animal}, \quad \text{Winged-Animal} \mid\!\sim \text{Flies}, \quad \text{Mammal} \mid\!\sim \neg\text{Flies} \}$$

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# Summary

- Instead of *ad hoc* extensions of the logical machinery, analyze the properties of nonmonotonic consequence relations.
- Correspondence between rule system and models for System C, and for System P also wrt a probabilistic semantics.
- Irrelevant information poses a problem. Solution approaches: rational monotony, maximum entropy, ...

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