Conflicts between defaults in Default Logic lead to multiple extensions.
Each extension corresponds to a maximal set of non-violated defaults.
Reasoning with defaults can also be achieved by a simpler mechanism: predicate or propositional logic + minimize the number of cases where a default (expressed as a conventional formula) is violated $\iff$ minimal models.
Notion of minimality: cardinality vs. set-inclusion.
Entailment with respect to Minimal Models

**Definition**

Let $A$ be a set of atomic propositions. Let $\Phi$ be a set of propositional formulae on $A$, and $B \subseteq A$ a set of abnormalities.

Then $\Phi \models_B \psi$ (\(\psi\ B\)-minimally follows from $\Phi$) if $\mathcal{I} \models \psi$ for all interpretations $\mathcal{I}$ such that $\mathcal{I} \models \Phi$ and there is no $\mathcal{I}'$ such that $\mathcal{I}' \models \Phi$ and $\{b \in B | \mathcal{I}' \models b\} \subset \{b \in B | \mathcal{I} \models b\}$. 
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Minimal models: example

\[ \Phi = \begin{cases} 
\text{student} \land \neg\text{ABstudent} \rightarrow \neg\text{earnsmoney}, & \text{student}, \\
\text{adult} \land \neg\text{ABadult} \rightarrow \text{earnsmoney}, & \text{student} \rightarrow \text{adult} 
\end{cases} \]

\( \Phi \) has the following models.

\[ \mathcal{I}_1 \models \text{student} \land \text{adult} \land \text{earnsmoney} \land \text{ABstudent} \land \text{ABadult} \]
\[ \mathcal{I}_2 \models \text{student} \land \text{adult} \land \neg\text{earnsmoney} \land \text{ABstudent} \land \text{ABadult} \]
\[ \mathcal{I}_3 \models \text{student} \land \text{adult} \land \text{earnsmoney} \land \text{ABstudent} \land \neg\text{ABadult} \]
\[ \mathcal{I}_4 \models \text{student} \land \text{adult} \land \neg\text{earnsmoney} \land \neg\text{ABstudent} \land \text{ABadult} \]
We can embed propositional minimal model reasoning in the propositional Default Logic.

**Theorem**

Let $A$ be a set of atomic propositions. Let $\Phi$ be a set of propositional formulae on $A$, and $B \subseteq A$. Then $\Phi \models_B \psi$ if and only if $\psi$ follows from $\langle D, W \rangle$ skeptically, where

$$D = \left\{ \frac{\neg b}{\neg b} \mid b \in B \right\} \text{ and } W = \Phi.$$
Proof sketch.

⇒ Assume there is extension $E$ of $\langle D, W \rangle$ such that $\psi \not\in E$. Hence there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models E$ and $\mathcal{I} \models \neg \psi$.

By the fact that there is no extension $F$ such that $E \subset F$, $\mathcal{I}$ is a $B$-minimal model of $\Phi$. Hence $\psi$ does not $B$-minimally follow from $\Phi$.

⇐ Assume $\psi$ does not $B$-minimally follow from $\Phi$. Hence there is an $B$-minimal model $\mathcal{I}$ of $\Phi$ such that $\mathcal{I} \not\models \psi$. Define $E = \text{Th}(\Phi \cup \{\neg b | b \in B, \mathcal{I} \models \neg b\})$. Now $\mathcal{I} \models E$ and because $\mathcal{I} \not\models \psi$, $\psi \not\in E$.

We can show that $E$ is an extension of $\langle D, W \rangle$. Because there is extension $E$ such that $\psi \not\in E$, $\psi$ does not skeptically follow from $\langle D, W \rangle$.  

$\Box$
**Relation to Default Logic: Proof**

**Proof sketch.**

⇒ Assume there is extension $E$ of $\langle D, W \rangle$ such that $\psi \notin E$. Hence there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models E$ and $\mathcal{I} \models \neg \psi$.

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 análisis de la imagen: La imagen muestra una página de un documento en el que se explora el tema de la reasoning no monotónica, específicamente la razón mínima (Minimal Model Reasoning). Se analiza la definición (Definition) y se proporciona un ejemplo (Example) relacionado con la embebida en el DL (DL Embedding). Este documento también discute el tema de la relation to Default logic: Proof (Relación a la lógica de defecto: Prueba).

La parte del documento que se muestra explora el concepto de prueba (Proof) en relación con la lógica no monotónica. Se presenta un esquema de prueba (Proof sketch) que incluye dos direcciones: la implicación (⇒) y el inverso (⇐).

Para la implicación (⇒), se asume que existe una extensión $E$ de $\langle D, W \rangle$ tal que $\psi \notin E$. Se deduce que existe un intepretación $\mathcal{I}$ tal que $\mathcal{I} \models E$ y $\mathcal{I} \models \neg \psi$. Esto implica que no existe una extensión $F$ tal que $E \subset F$, por lo que $\mathcal{I}$ es un modelo mínimo de $\Phi$. En consecuencia, $\psi$ no se deduce mínimamente de $\Phi$.

Para el inverso (⇐), se asume que $\psi$ no se deduce mínimamente de $\Phi$. Se concluye que existe un modelo mínimo $\mathcal{I}$ de $\Phi$ tal que $\mathcal{I} \not\models \psi$. Se define $E = \text{Th}(\Phi \cup \{\neg b | b \in B, \mathcal{I} \models \neg b\})$. Se demuestra que $\mathcal{I} \models E$ y porque $\mathcal{I} \not\models \psi$, $\psi \notin E$. Entonces, se puede mostrar que $E$ es una extensión de $\langle D, W \rangle$. Dado que existe una extensión $E$ tal que $\psi \notin E$, $\psi$ no se deriva de forma mínima de $\langle D, W \rangle$. 

Es importante destacar que el texto se presenta sin cambios de ortografía o gramática, manteniendo la precisión y coherencia del contenido original.
Nonmonotonic Logic Programs: Background

- **Answer set semantics**: a formalization of negation-as-failure in logic programming (Prolog)
- Other formalizations: well-founded semantics, perfect-model semantics, inflationary semantics, ...
- Can be viewed as a simpler variant of default logic.
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Nonmonotonic Logic Programs

- Rules $c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k$
  where $\{c, b_1, \ldots, b_m, d_1, \ldots, d_k\} \subseteq A$ for a set $A = \{a_1, \ldots, a_n\}$ of propositions.

- Meaning similar to default logic: If
  1. we have derived $b_1, \ldots, b_m$ and
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then derive $c$.

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Answer Sets – Formal Definition

- **Reduct** $P^\Delta$ of a program $P$ with respect to a set of atoms $\Delta \subseteq A$:

\[
\{ c \leftarrow b_1, \ldots, b_m \mid (c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k) \in P, \{d_1, \ldots, d_k\} \cap \Delta = \emptyset \}
\]

- **Closure** $\text{dcl}(P) \subseteq A$ of a set $P$ of rules without $\text{not}$ is defined by iterative application of the rules in the obvious way.

- A set of propositions $\Delta \subseteq A$ is an answer set of $P$ iff $\Delta = \text{dcl}(P^\Delta)$.
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Examples

- $P_1 = \{ a \leftarrow, \ b \leftarrow a, \ c \leftarrow b \}$
- $P_2 = \{ a \leftarrow b, \ b \leftarrow a \}$
- $P_3 = \{ p \leftarrow \text{not } p \}$
- $P_4 = \{ p \leftarrow \text{not } q, \ q \leftarrow \text{not } p \}$
- $P_5 = \{ p \leftarrow \text{not } q, \ q \leftarrow \text{not } p, \ \leftarrow p \}$
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Complexity: existence of answer sets is NP-complete

1. **Membership in NP:** Guess $\Delta \subseteq A$ (nondet. polytime), compute $P^\Delta$, compute its closure, compare to $\Delta$ (everything det. polytime).

2. **NP-hardness:** Reduction from 3SAT: an answer set exists iff clauses are satisfiable:

   $p \leftarrow \neg \hat{p}$
   $\hat{p} \leftarrow \neg p$

   for every proposition $p$ occurring in the clauses, and

   $\leftarrow \neg l_1', \neg l_2', \neg l_3'$

   for every clause $l_1 \lor l_2 \lor l_3$, where $l_i' = p$ if $l_i = p$ and $l_i' = \hat{p}$ if $l_i = \neg p$. 
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Programs for Reasoning with Answer Sets

- smodels (Niemelä & Simons), dlv (Eiter et al.), ...

- Schematic input:

```prolog
p(X) :- not q(X).
q(X) :- not p(X).
r(a).
r(b).
r(c).
anc(X,Y) :- par(X,Y).
anc(X,Y) :- par(X,Z), anc(Z,Y).
par(a,b). par(a,c). par(b,d).
female(a).
male(X) :- not(female(X)).
forefather(X,Y) :-
    anc(X,Y), male(X).
```

Difference to the Propositional Logic

- The ancestor relation is the transitive closure of the parent relation.
- Transitive closure cannot be (concisely) represented in propositional/predicate logic.

\[
\begin{align*}
par(X,Y) \rightarrow anc(X,Y) \\
par(X,Z) \land anc(Z,Y) \rightarrow anc(X,Y)
\end{align*}
\]

The above formulae only guarantee that \textit{anc} is a superset of the transitive closure of \textit{par}.

- For transitive closure one needs the minimality condition in some form: nonmonotonic logics, fixpoint logics, ...
Stratification

The reason for multiple answer sets is the fact that $a$ may depend on $b$ and simultaneously $b$ may depend on $a$. The lack of this kind of circular dependencies makes reasoning easier.

**Definition**

A logic program $P$ is **stratified** if $P$ can be partitioned to $P = P_1 \cup \cdots \cup P_n$ so that for all $i \in \{1, \ldots, n\}$ and $(c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k) \in P_i$,

1. there is no not $c$ in $P_i$ and
2. there are no occurrences of $c$ anywhere in $P_1 \cup \cdots \cup P_{i-1}$. 
Theorem

A stratified program $P$ has exactly one answer set. The unique answer set can be computed in polynomial time.

Example

Our earlier examples with more than one or no answer sets:

$$P_3 = \{ p \leftarrow \text{not} \ p \}$$

$$P_4 = \{ p \leftarrow \text{not} \ q, \ q \leftarrow \text{not} \ p \}$$
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Applications of Logic Programs

1. Simple forms of default reasoning (inheritance networks)

2. A solution to the frame problem: instead of using frame axioms, use defaults

\[ a_{t+1} \leftarrow a_t, \text{not} \neg a_{t+1} \]

By default, truth-values of facts stay the same.

3. deductive databases (Datalog\(\neg\))

4. et cetera: Everything that can be done with propositional logic can also be done with propositional nonmonotonic logic programs.
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By default, truth-values of facts stay the same.

3. Deductive databases (Datalog\(^\neg\))

4. Et cetera: Everything that can be done with propositional logic can also be done with propositional nonmonotonic logic programs.
Applications of Logic Programs

1. Simple forms of default reasoning (inheritance networks)

2. A solution to the frame problem: instead of using frame axioms, use defaults

\[ a_{t+1} \leftarrow a_t, \text{not } \neg a_{t+1} \]

By default, truth-values of facts stay the same.

3. deductive databases (Datalog\(\neg\))

4. et cetera: Everything that can be done with propositional logic can also be done with propositional nonmonotonic logic programs.
