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First Order Logic

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Why First-Order Logic (FOL)?

- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - *All CS students know formal logic*
  - *Peter is a CS student*
  - Therefore, *Peter knows formal logic*
  - Not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.

Syntax

- variable symbols: $x, y, z, \ldots$
- n-ary function symbols: $f(\cdots), g(\cdots), \ldots$
- constant symbols: $a, b, c, \ldots$
- n-ary predicate symbols: $P(\cdots), Q(\cdots), \ldots$

Terms	t	$\longrightarrow$	x	variable
		$\mid$	$f(t_1, \ldots, t_n)$	function application
		$\mid$	a	constant
Formulae	φ	$\longrightarrow$	$P(t_1, \ldots, t_n)$	atomic formula
		$\mid$	$\ldots$	propositional connectives
		$\mid$	$\forall x(\varphi')$	universal quantification
		$\mid$	$\exists x(\varphi')$	existential quantification

ground term, etc.: term, etc. without variable occurrences

Semantics: Idea

- In FOL, the universe of discourse consists of objects, functions over these objects, and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $\mathcal{I}(x)$ we write $x^{\mathcal{I}}$.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt all these universes.

Formal Semantics: Interpretations

Interpretations: $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{D}$ being an arbitrary non-empty set and $\mathcal{I}$ being a function which maps

- n-ary function symbols f to n-ary functions $f^{\mathcal{I}} \in [\mathcal{D}^n \to \mathcal{D}]$,
- constant symbols a to objects $a^{\mathcal{I}} \in \mathcal{D}$, and
- n-ary predicates P to n-ary relations $P^{\mathcal{I}} \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^{\mathcal{I}} = f^{\mathcal{I}}(t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}}) \ (\in \mathcal{D})$$

Truth of ground atoms:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I}}, \ldots, t_n{}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

Examples

$$\begin{aligned} \mathcal{D} &= \{d_1, \ldots, d_n\}, n \geq 2 \\ \mathrm{a}^{\mathcal{I}} &= d_1 \\ \mathrm{b}^{\mathcal{I}} &= d_2 \\ \mathrm{eye}^{\mathcal{I}} &= \{d_1\} \\ \mathrm{red}^{\mathcal{I}} &= \mathcal{D} \\ \mathcal{I} &\models \mathrm{red(b)} \\ \mathcal{I} &\not\models \mathrm{eye(b)} \end{aligned}$$

$$\begin{aligned} \mathcal{D} &= \{1, 2, 3, \ldots\} \\ 1^{\mathcal{I}} &= 1 \\ 2^{\mathcal{I}} &= 2 \\ &\vdots \\ \mathsf{even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\ \mathsf{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\ \mathcal{I} &\not\models \mathsf{even}(3) \\ \mathcal{I} &\models \mathsf{even}(\mathsf{succ}(3)) \end{aligned}$$

Formal Semantics: Variable Maps

V is the set of variables. Function $\alpha\colon V \to \mathcal{D}$ is a variable map.
Notation: $\alpha[x/d]$ is identical to α except for x where $\alpha[x/d](x) = d$.
Interpretation of terms under $\mathcal{I}, \alpha$:

$$\begin{aligned} x^{\mathcal{I},\alpha} &= \alpha(x) \\ a^{\mathcal{I},\alpha} &= a^{\mathcal{I}} \\ (f(t_1, \ldots, t_n))^{\mathcal{I},\alpha} &= f^{\mathcal{I}}(t_1{}^{\mathcal{I},\alpha}, \ldots, t_n{}^{\mathcal{I},\alpha}) \end{aligned}$$

Truth of atomic formulae:

$$\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I},\alpha}, \ldots, t_n{}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}}$$

Example (cont.):
$\alpha = \{x \mapsto d_1, y \mapsto d_2\}$ $\quad \mathcal{I}, \alpha \models \mathsf{red}(x)$ $\quad \mathcal{I}, \alpha[y/d_1] \models \mathsf{eye}(y)$

Formal Semantics: Truth

Truth of φ by $\mathcal{I}$ under α ($\mathcal{I}, \alpha \models \varphi$) is defined as follows.

$$\begin{aligned} \mathcal{I}, \alpha \models P(t_1, \ldots, t_n) &\quad \text{iff} \quad \langle t_1{}^{\mathcal{I},\alpha}, \ldots, t_n{}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}} \\ \mathcal{I}, \alpha \models \neg\varphi &\quad \text{iff} \quad \mathcal{I}, \alpha \not\models \varphi \\ \mathcal{I}, \alpha \models \varphi \wedge \psi &\quad \text{iff} \quad \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\ \mathcal{I}, \alpha \models \varphi \vee \psi &\quad \text{iff} \quad \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \\ \mathcal{I}, \alpha \models \varphi \rightarrow \psi &\quad \text{iff} \quad \text{if } \mathcal{I}, \alpha \models \varphi, \text{then } \mathcal{I}, \alpha \models \psi \\ \mathcal{I}, \alpha \models \varphi \leftrightarrow \psi &\quad \text{iff} \quad \mathcal{I}, \alpha \models \varphi, \text{ iff } \mathcal{I}, \alpha \models \psi \\ \mathcal{I}, \alpha \models \forall x\, \varphi &\quad \text{iff} \quad \mathcal{I}, \alpha[x/d] \models \varphi \text{ for all } d \in \mathcal{D} \\ \mathcal{I}, \alpha \models \exists x\, \varphi &\quad \text{iff} \quad \mathcal{I}, \alpha[x/d] \models \varphi \text{ for some } d \in \mathcal{D} \end{aligned}$$

Examples

$$\Theta = \left\{ \begin{array}{l} \text{eye}(a), \text{eye}(b) \\ \forall x(\text{eye}(x) \rightarrow \text{red}(x)) \end{array} \right\}$$
$$\mathcal{D} = \{d_1, \ldots, d_n,\} \; n > 1$$
$$a^{\mathcal{I}} = d_1$$
$$b^{\mathcal{I}} = d_1$$
$$\text{eye}^{\mathcal{I}} = \{d_1\}$$
$$\text{red}^{\mathcal{I}} = \mathcal{D}$$
$$\alpha = \{(x \mapsto d_1), (y \mapsto d_2)\}$$

Questions:

$\mathcal{I}, \alpha \models \text{eye}(b) \vee \neg\text{eye}(b)$? *Yes*

$\mathcal{I}, \alpha \models \text{eye}(x) \rightarrow \text{eye}(x) \vee \text{eye}(y)$? *Yes*

$\mathcal{I}, \alpha \models \text{eye}(x) \rightarrow \text{eye}(y)$? *No*

$\mathcal{I}, \alpha \models \text{eye}(a) \wedge \text{eye}(b)$? *Yes*

$\mathcal{I}, \alpha \models \forall x(\text{eye}(x) \rightarrow \text{red}(x))$? *Yes*

$\mathcal{I}, \alpha \models \Theta$? *Yes*

Terminology

$\mathcal{I}, \alpha$ is a model of φ iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be satisfiable, unsatisfiable, falsifiable, valid.
Two formulae φ and ψ are logically equivalent ($\varphi \equiv \psi$) iff for all $\mathcal{I}, \alpha$:

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

Note: P(x) $\not\equiv$ P(y)!
Logical Implication is also similar to propositional logic:

$$\Theta \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

Free and Bound Variables

Variables can be free or bound (by a quantifier) in a formula:

$$\text{free}(x) = \{x\}$$
$$\text{free}(f(t_1, \ldots, t_n)) = \text{free}(t_1) \cup \ldots \cup \text{free}(t_n)$$
$$\text{free}(P(t_1, \ldots, t_n)) = \text{free}(t_1) \cup \ldots \cup \text{free}(t_n)$$
$$\text{free}(\neg\varphi) = \text{free}(\varphi)$$
$$\text{free}(\varphi * \psi) = \text{free}(\varphi) \cup \text{free}(\psi) \quad * = \vee, \wedge, \rightarrow, \leftrightarrow$$
$$\text{free}(\Xi x \varphi) = \text{free}(\varphi) - \{x\} \quad \Xi = \forall, \exists$$

Example: $\forall$x (R($\boxed{\text{y}}$,$\boxed{\text{z}}$) $\wedge$ $\exists$ y ($\neg$P(y,x) $\vee$ R(y,$\boxed{\text{z}}$)))
Framed occurrences are free, all others are bound.

Open & Closed Formulae

- Formulae without free variables are called closed formulae or sentences. Formulae with free variables are called open formulae.
- Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of $\forall$ and $\exists$).
- Note that *logical equivalence*, *satisfiability*, and entailment are independent from variable maps if we consider only closed formulae.
- For closed formulae, we omit α in connection with $\models$:

$$\mathcal{I} \models \varphi.$$

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