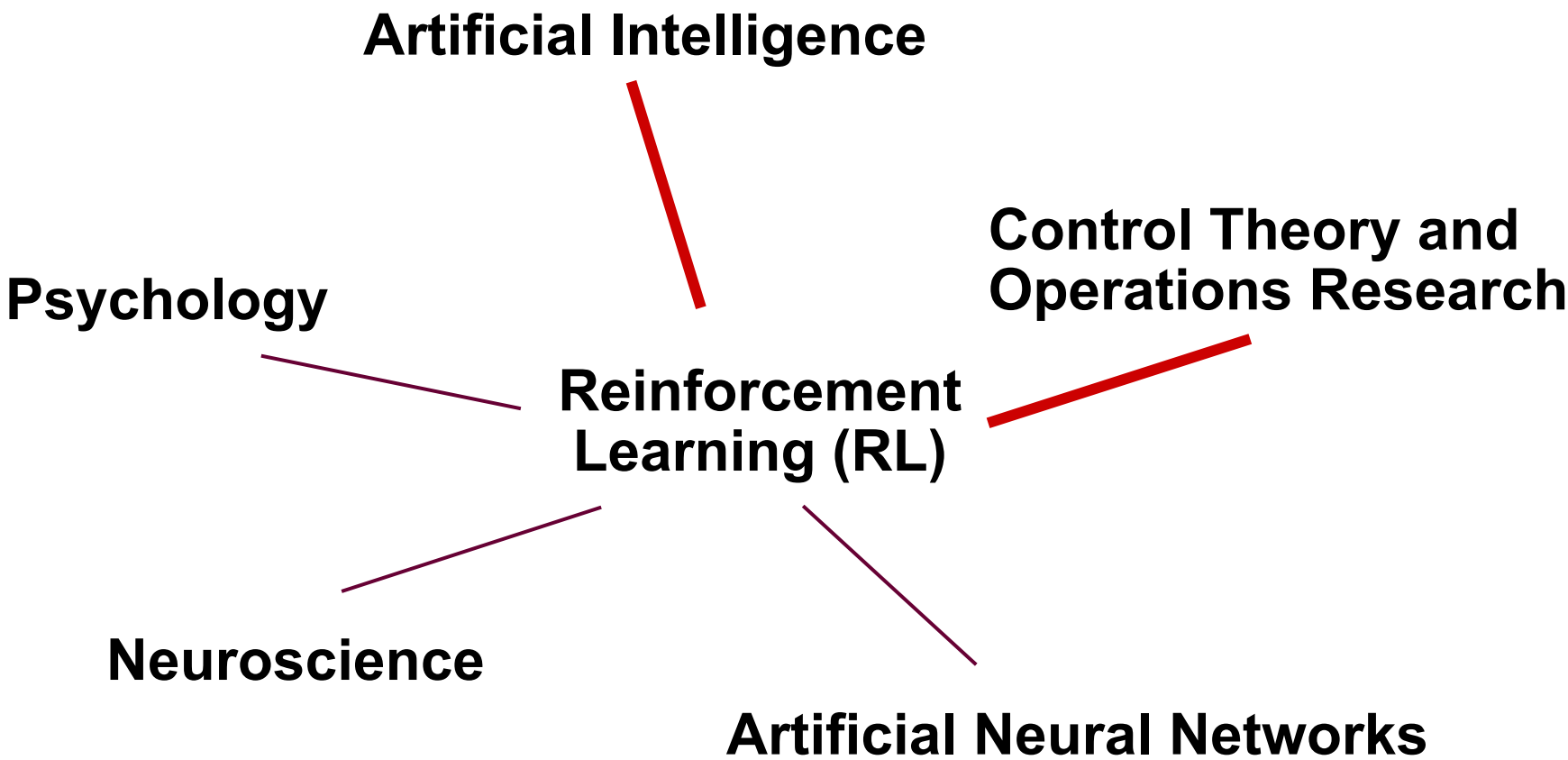
Reinforcement Learning

- Mainly based on "Reinforcement Learning – An Introduction" by Richard Sutton and Andrew Barto
- Slides are mainly based on the course material provided by the same authors

http://www.cs.ualberta.ca/~sutton/book/the-book.html

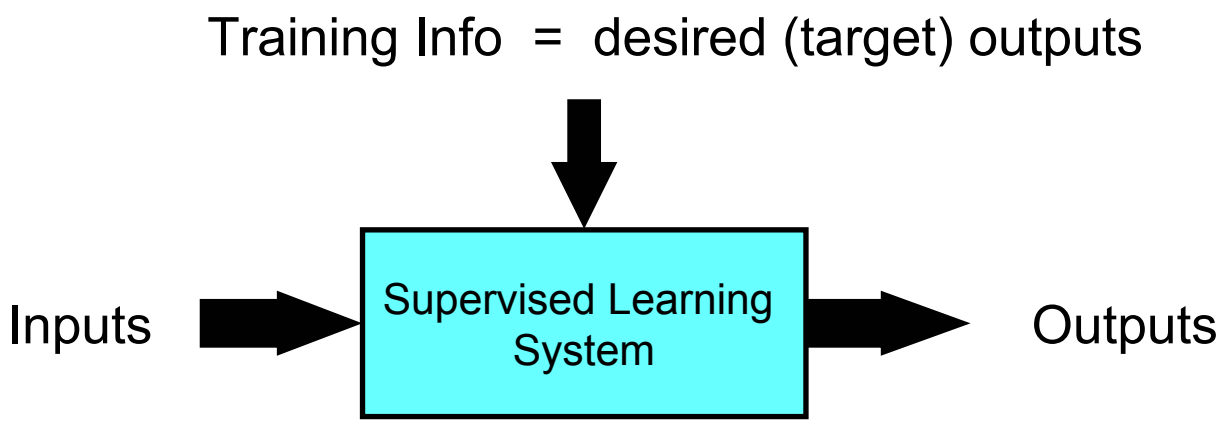
Learning from Experience Plays a Role in …



What is Reinforcement Learning?

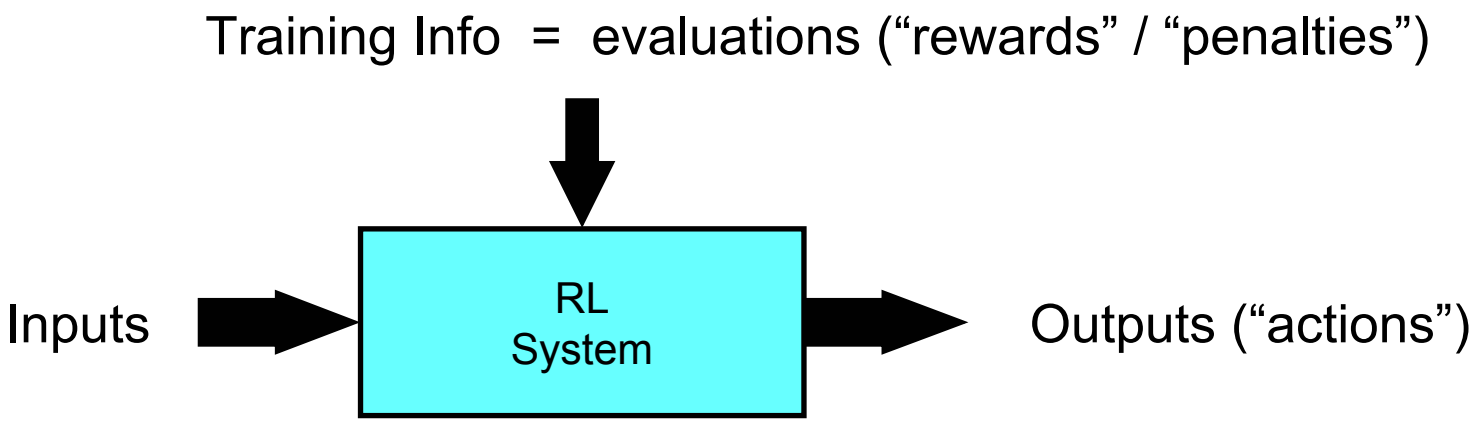
- Learning from interaction
- Goal-oriented learning
- Learning about, from, and while interacting with an external environment
- Learning what to do—how to map situations to actions—so as to maximize a numerical reward signal

Supervised Learning



Error = (target output – actual output)

Reinforcement Learning



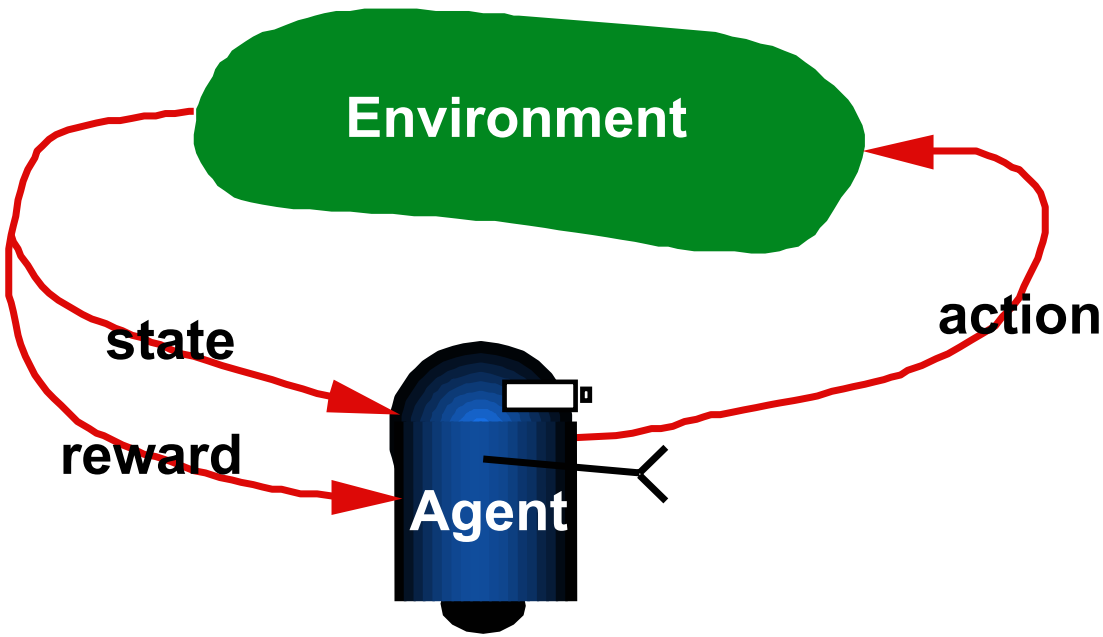
Objective: get as much reward as possible

Key Features of RL

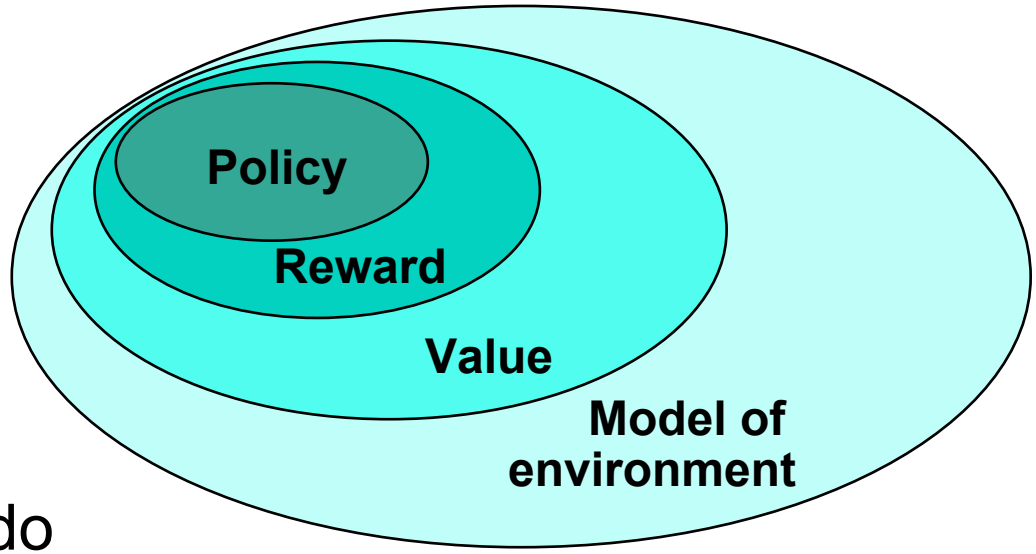
- Learner is not told which actions to take
- Trial-and-Error search
- Possibility of delayed reward (sacrifice short-term gains for greater long-term gains)
- The need to ***explore*** and ***exploit***
- Considers the whole problem of a goal-directed agent interacting with an uncertain environment

Complete Agent

- Temporally situated
- Continual learning and planning
- Object is to ***affect*** the environment
- Environment is stochastic and uncertain

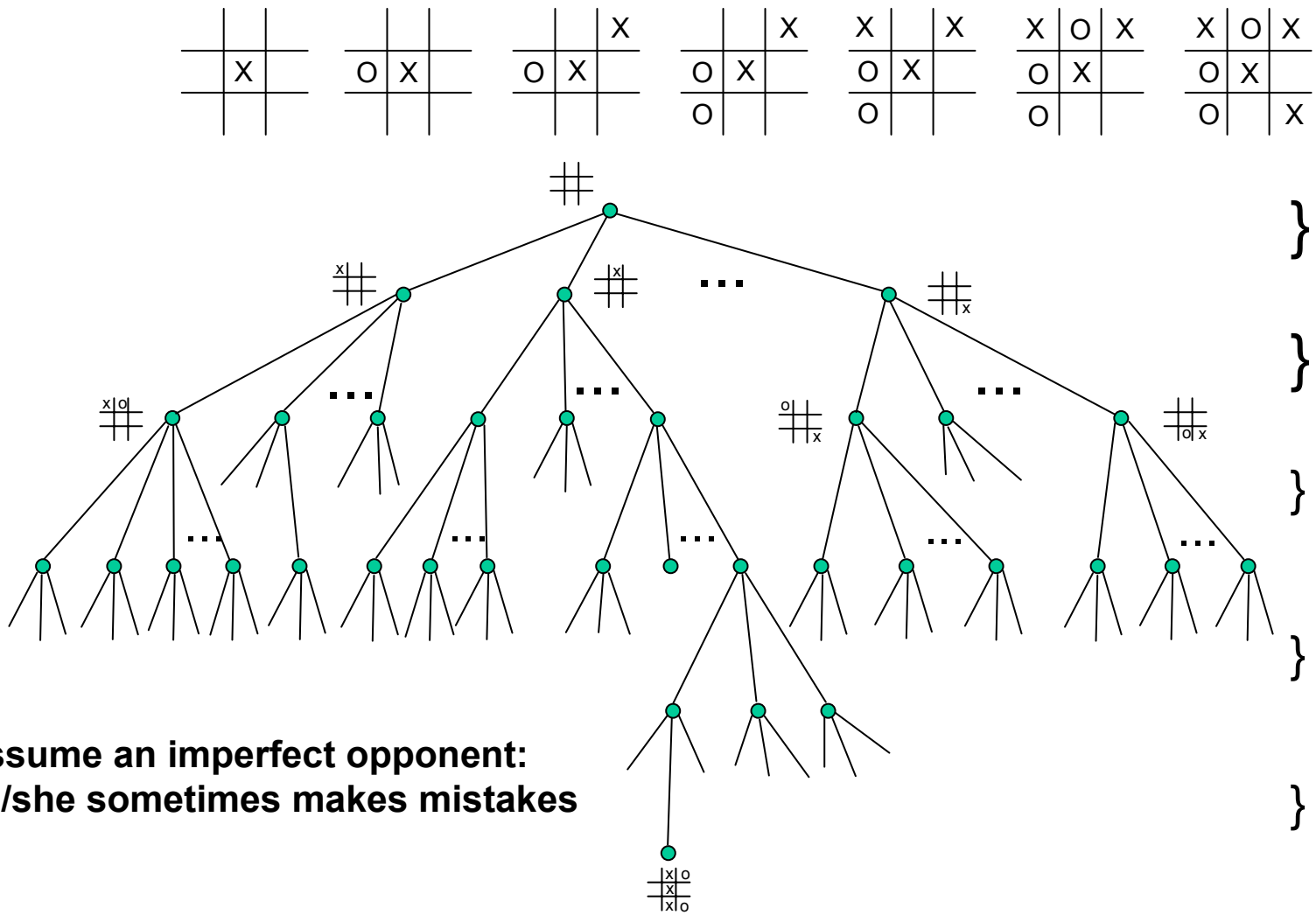


Elements of RL



- **Policy**: what to do
- **Reward**: what is good
- **Value**: what is good because it *predicts* reward
- **Model**: what follows what

An Extended Example: Tic-Tac-Toe



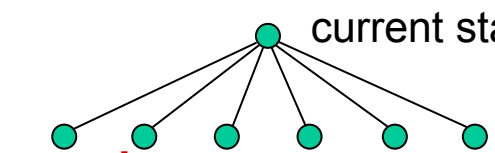
Assume an imperfect opponent:
he/she sometimes makes mistakes

An RL Approach to Tic-Tac-Toe

1. Make a table with one entry per state:

State	$V(s)$ – estimated probability of winning	
	.5	?
	.5	?
⋮	⋮	
	1	win
⋮	⋮	
	0	loss
⋮	⋮	
	0	draw

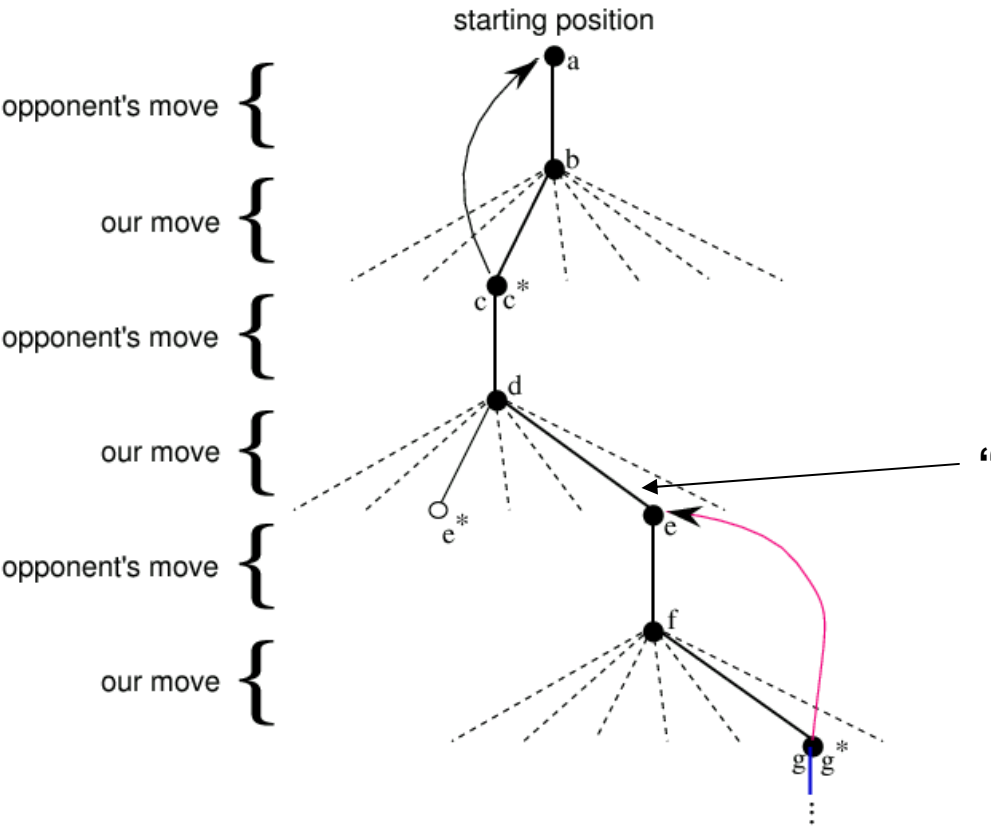
2. Now play lots of games. To pick our moves, look ahead one step:



Just pick the next state with the highest estimated prob. of winning — the largest $V(s)$; a ***greedy*** move.

But 10% of the time pick a move at random; an ***exploratory move***.

RL Learning Rule for Tic-Tac-Toe



"Exploratory" move

s – *the state before our greedy move*

s' – *the state after our greedy move*

We increment each $V(s)$ *toward* $V(s')$ – *a* ***backup*** *:*

$$V(s) \leftarrow V(s) + \alpha\left[V(s') - V(s)\right]$$

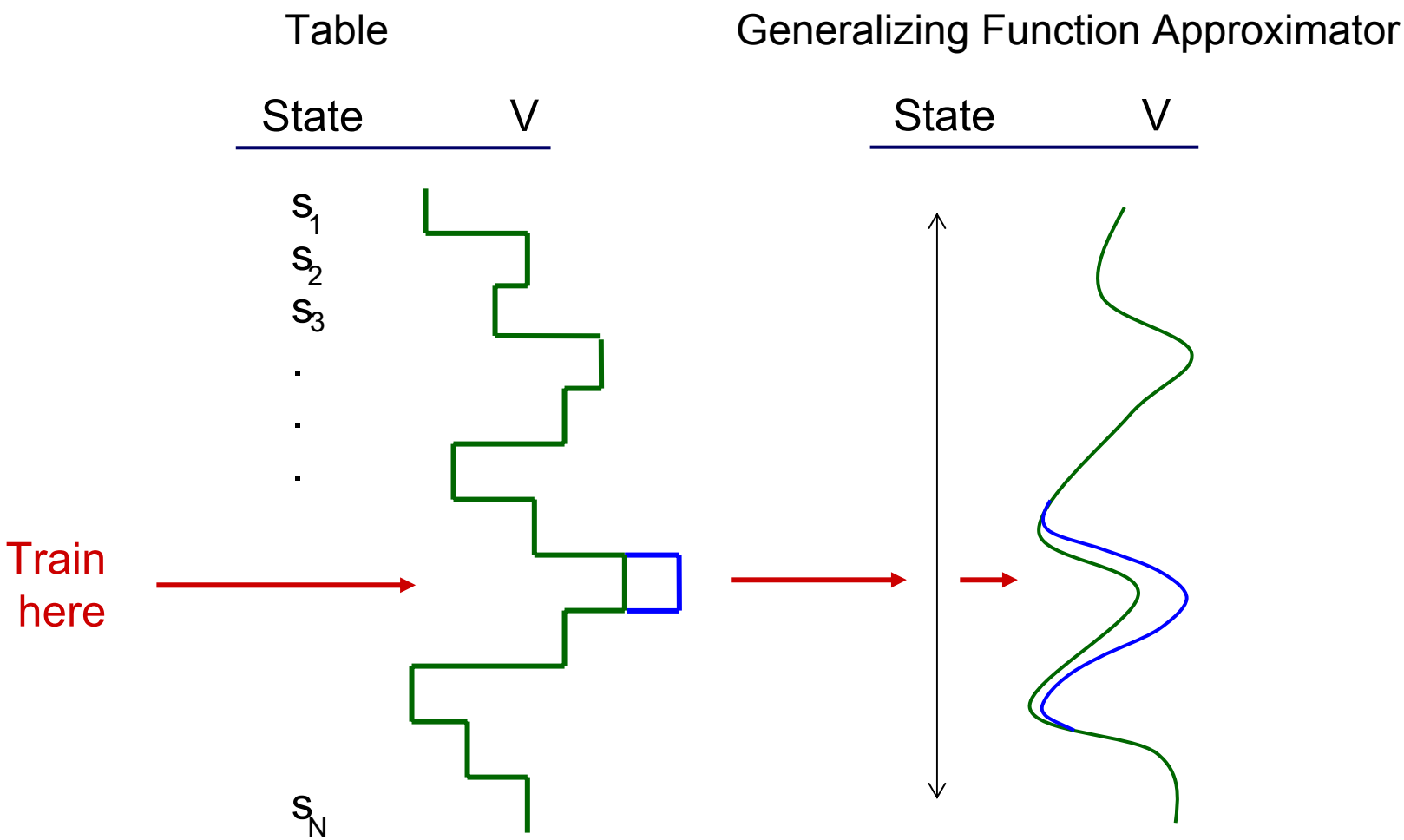
a small positive fraction, e.g., $\alpha = .1$

the ***step-size parameter***

How can we improve this T.T.T. player?

- Take advantage of symmetries
  - representation/generalization
  - How might this backfire?
- Do we need “random” moves? Why?
  - Do we always need a full 10%?
- Can we learn from “random” moves?
- Can we learn offline?
  - Pre-training from self play?
  - Using learned models of opponent?
- . . .

e.g. Generalization



How is Tic-Tac-Toe Too Easy?

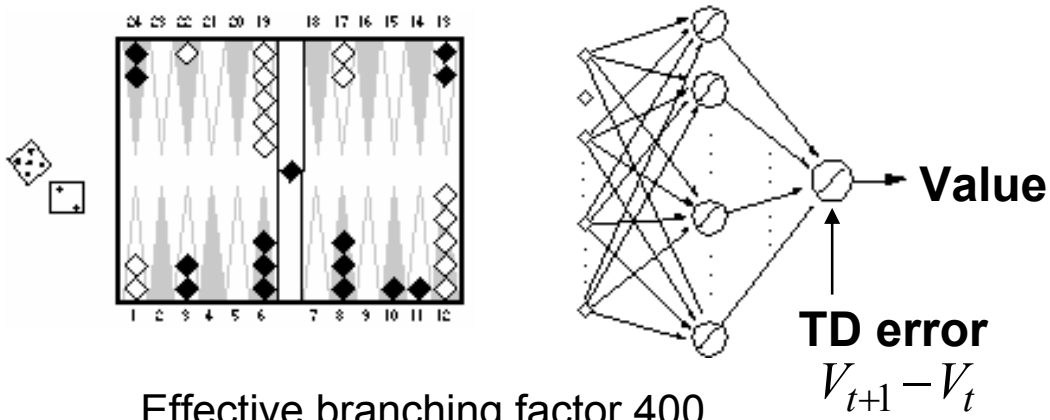
- Finite, small number of states
- One-step look-ahead is always possible
- State completely observable
- …

Some Notable RL Applications

- **TD-Gammon**: Tesauro
  - world's best backgammon program
- **Elevator Control**: Crites & Barto
  - high performance down-peak elevator controller
- **Dynamic Channel Assignment**: Singh & Bertsekas, Nie & Haykin
  - high performance assignment of radio channels to mobile telephone calls
- …

TD-Gammon

Tesauro, 1992–1995



Action selection by 2–3 ply search

Effective branching factor 400

Start with a random network

Play very many games against self

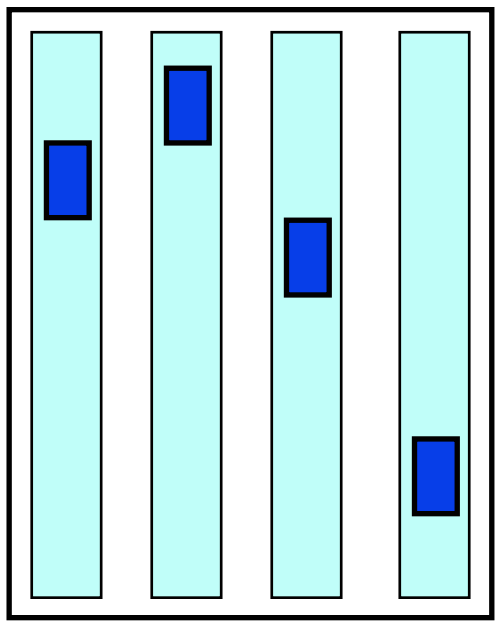
Learn a value function from this simulated experience

This produces arguably the best player in the world

Elevator Dispatching

Crites and Barto, 1996

10 floors, 4 elevator cars



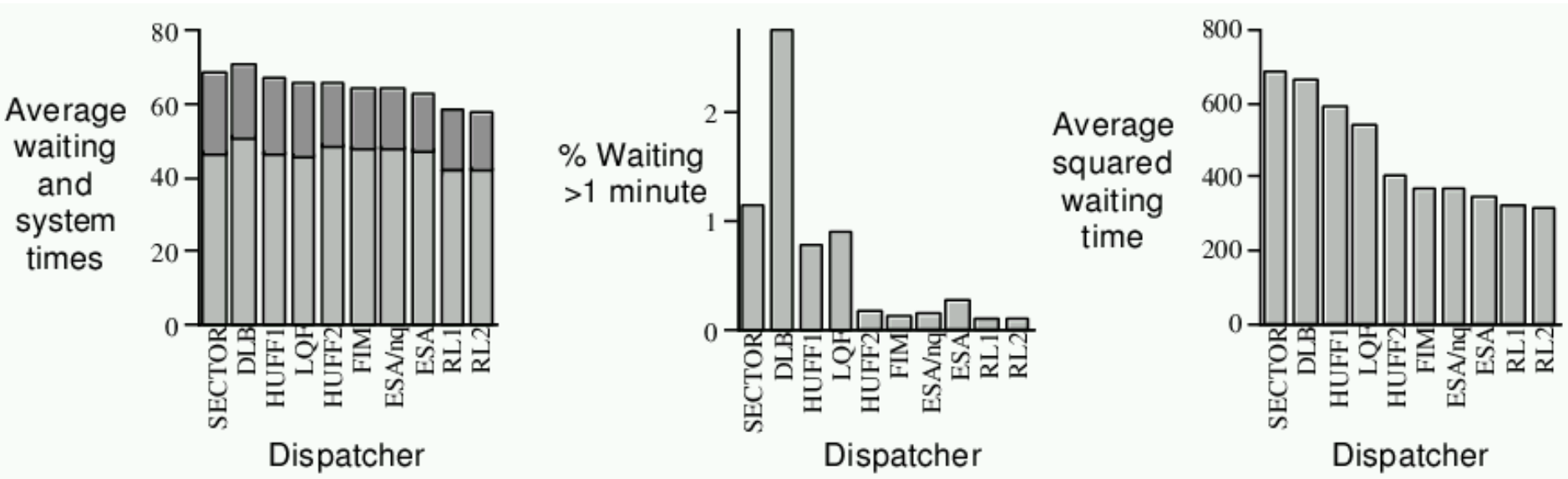
STATES: button states; positions, directions, and motion states of cars; passengers in cars & in halls

ACTIONS: stop at, or go by, next floor

REWARDS: roughly, –1 per time step for each person waiting

Conservatively about 10^{22} states

Performance Comparison



Evaluative Feedback

- **Evaluating** actions vs. **instructing** by giving correct actions
- Pure evaluative feedback depends totally on the action taken. Pure instructive feedback depends not at all on the action taken.
- Supervised learning is instructive; optimization is evaluative
- **Associative** vs. **Nonassociative**:
  - Associative: inputs mapped to outputs; learn the best output **for each** input
  - Nonassociative: “learn” (find) one best output
- *n*-armed bandit (at least how we treat it) is:
  - Nonassociative
  - Evaluative feedback

The *n*-Armed Bandit Problem

- Choose repeatedly from one of *n* actions; each choice is called a **play**
- After each play a_t, you get a reward r_t, where

$$E\left\langle r_t \mid a_t \right\rangle = Q^*(a_t)$$

These are unknown **action values**
Distribution of r_t depends only on a_t

- Objective is to maximize the reward in the long term, e.g., over 1000 plays

To solve the *n*-armed bandit problem, you must **explore** a variety of actions and then **exploit** the best of them.

The Exploration/Exploitation Dilemma

- Suppose you form estimates

$$Q_t(a) \approx Q^*(a)$$ **action value estimates**

- The **greedy** action at *t* is

$$a_t^* = \arg\max_a Q_t(a)$$

$$a_t = a_t^* \Rightarrow \text{exploitation}$$

$$a_t \neq a_t^* \Rightarrow \text{exploration}$$

- You can't exploit all the time; you can't explore all the time
- You can never stop exploring; but you should always reduce exploring

Action-Value Methods

- Methods that adapt action-value estimates and nothing else, e.g.: suppose by the *t*-th play, action a had been chosen k_a times, producing rewards $r_1, r_2, \ldots, r_{k_a}$, then

$$Q_t(a) = \frac{r_1 + r_2 + \cdots + r_{k_a}}{k_a}$$ "sample average"

$$\lim_{k_a \to \infty} Q_t(a) = Q^*(a)$$

ε-Greedy Action Selection

- Greedy action selection:

$$a_t = a_t^* = \arg\max_a Q_t(a)$$

- ε-Greedy:

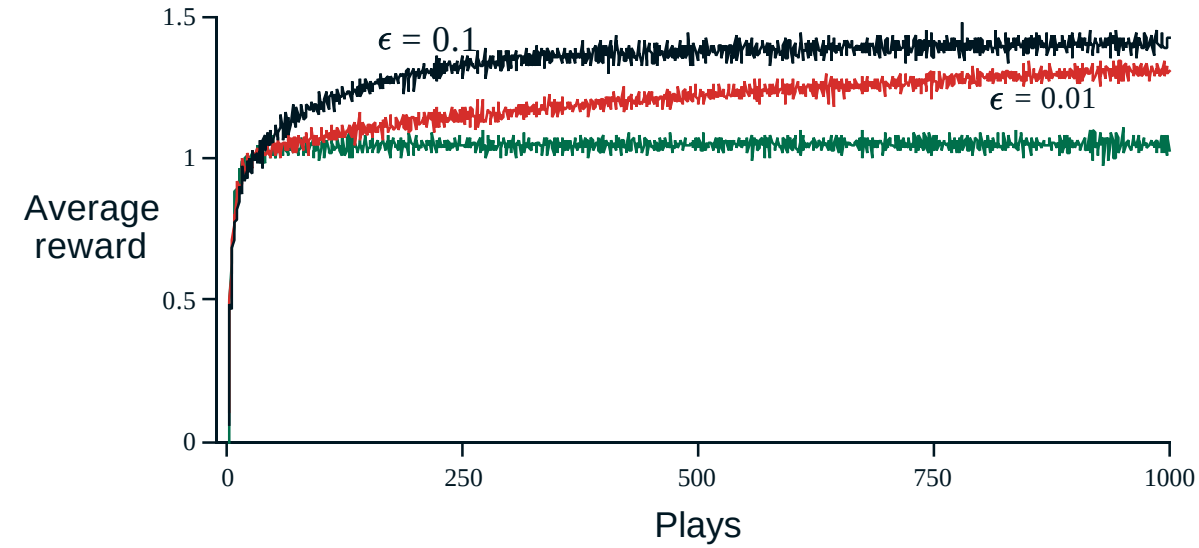
$$a_t = \begin{cases} a_t^* & \text{with probability } 1-\varepsilon \\ \text{random action with probability } \varepsilon \end{cases}$$

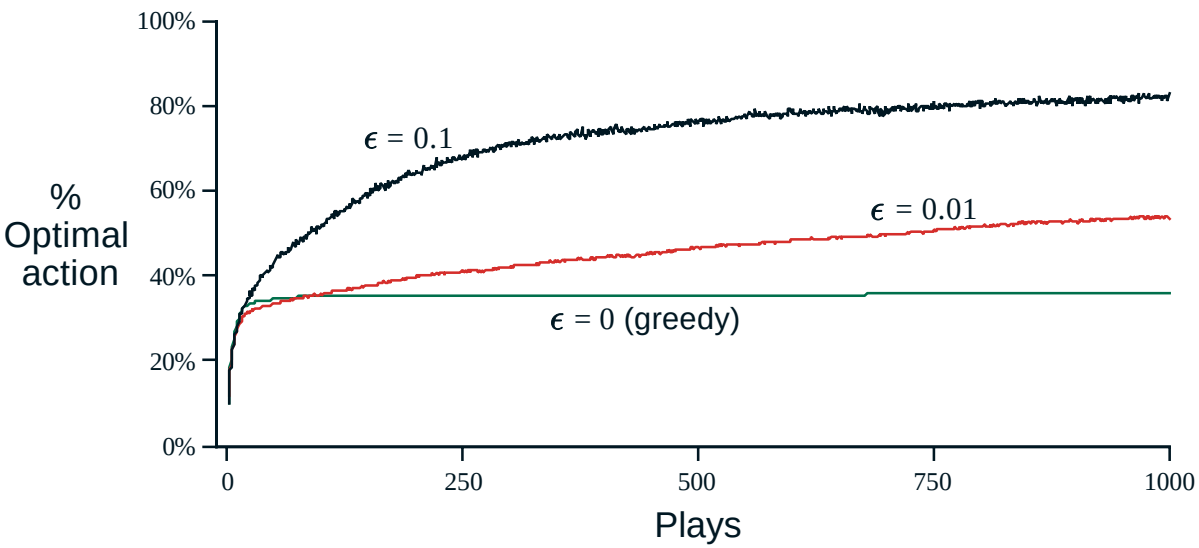
... the simplest way to try to balance exploration and exploitation

10-Armed Testbed

- n = 10 possible actions
- Each $Q^*(a)$ is chosen randomly from a normal distribution: $N(0,1)$
- each r_t is also normal: $N(Q^*(a_t),1)$
- 1000 plays
- repeat the whole thing 2000 times and average the results
- *Evaluative versus instructive feedback*

ε-Greedy Methods on the 10-Armed Testbed





Softmax Action Selection

- Softmax action selection methods grade action probs. by estimated values.
- The most common softmax uses a Gibbs, or Boltzmann, distribution:

  Choose action a on play t with probability

$$\frac{e^{Q_t(a)/\tau}}{\sum_{b=1}^{n} e^{Q_t(b)/\tau}},$$

  where τ is the "computational temperature"

Evaluation Versus Instruction

- Suppose there are K possible actions and you select action number k.
- Evaluative feedback would give you a single score f, say 7.2.
- Instructive information, on the other hand, would say that action k', which is eventually different from action k, have actually been correct.
- Obviously, instructive feedback is much more informative, (even if it is noisy).

Binary Bandit Tasks

Suppose you have just **two** actions: $a_t = 1 \quad \text{or} \quad a_t = 2$

and just **two** rewards: $r_t = \mathit{success} \quad \text{or} \quad r_t = \mathit{failure}$

Then you might infer a **target** or **desired action**:

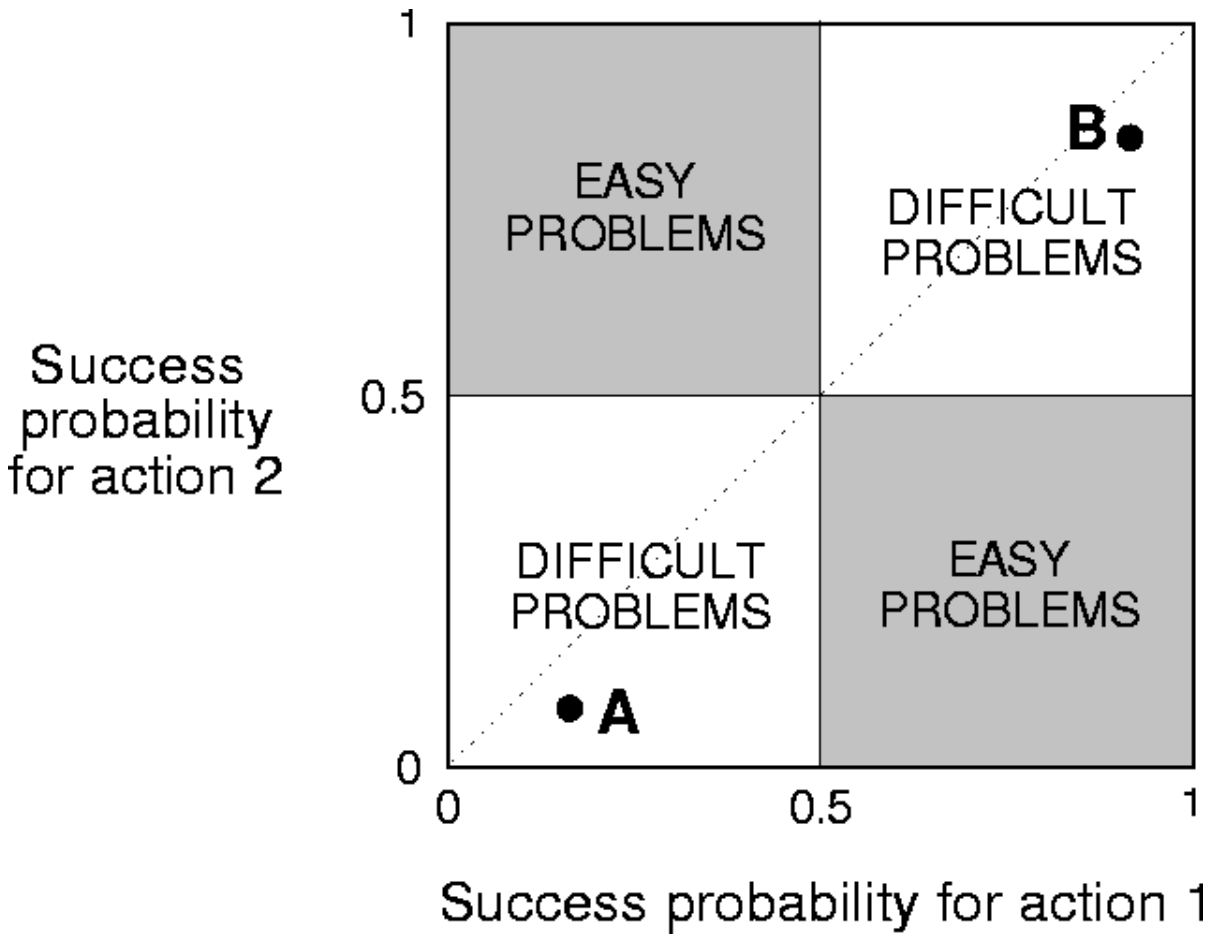
$$d_t = \begin{cases} a_t & \text{if } \mathit{success} \\ \text{the other action} & \text{if } \mathit{failure} \end{cases}$$

and then always play the action that was most often the target

Call this the **supervised algorithm**.
It works fine on deterministic tasks but is suboptimal if the rewards are stochastic.

Contingency Space

The space of all possible binary bandit tasks:



Linear Learning Automata

Let $\pi_t(a) = \Pr\{a_t = a\}$ be the only adapted parameter

L_{R-I} (Linear, reward - inaction)

On *success* : $\pi_{t+1}(a_t) = \pi_t(a_t) + \alpha(1 - \pi_t(a_t)) \qquad 0 < \alpha < 1$

(the other action probs. are adjusted to still sum to 1)

On *failure* : no change

L_{R-P} (Linear, reward - penalty)

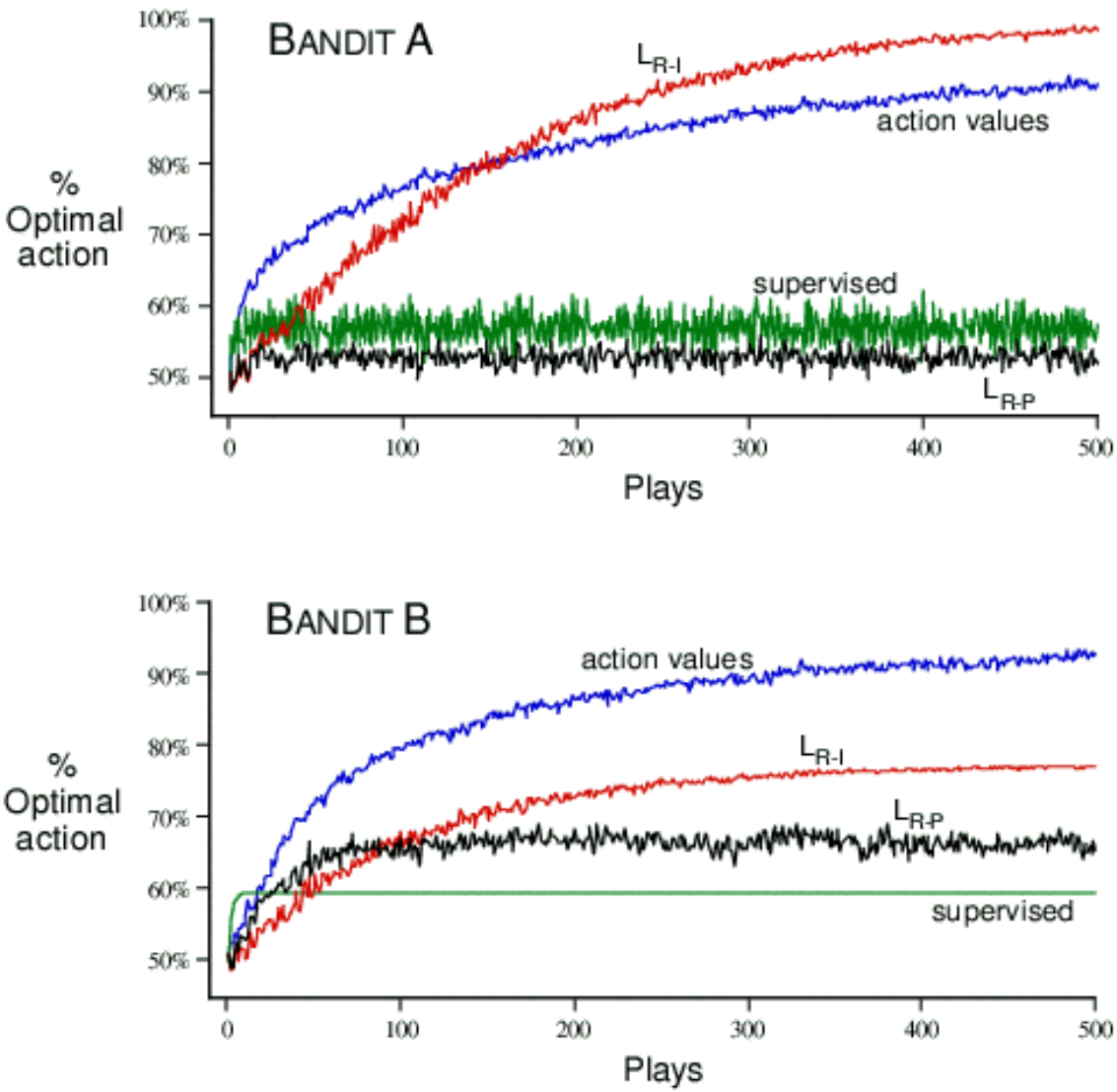
On *success* : $\pi_{t+1}(a_t) = \pi_t(a_t) + \alpha(1 - \pi_t(a_t)) \qquad 0 < \alpha < 1$

(the other action probs. are adjusted to still sum to 1)

On *failure* : $\pi_{t+1}(a_t) = \pi_t(a_t) + \alpha(0 - \pi_t(a_t)) \qquad 0 < \alpha < 1$

For two actions, a stochastic, incremental version of the supervised algorithm

Performance on Binary Bandit Tasks A and B



Incremental Implementation

Recall the sample average estimation method:

The average of the first k rewards is (dropping the dependence on a):

$$Q_k = \frac{r_1 + r_2 + \cdots r_k}{k}$$

Can we do this incrementally (without storing all the rewards)?

We could keep a running sum and count, or, equivalently:

$$Q_{k+1} = Q_k + \frac{1}{k+1}\left[r_{k+1} - Q_k\right]$$

This is a common form for update rules:

NewEstimate **=** *OldEstimate* **+** *StepSize*[*Target* **–** *OldEstimate*]

Computation

$$
\begin{aligned}
Q_{k+1} &= \frac{1}{k+1}\sum_{i=1}^{k+1} r_i \\
&= \frac{1}{k+1}\left(r_{k+1} + \sum_{i=1}^{k} r_i \right) \\
&= \frac{1}{k+1}\left(r_{k+1} + kQ_k + Q_k - Q_k \right) \\
&= Q_k + \frac{1}{k+1}\left[r_{k+1} - Q_k \right]
\end{aligned}
$$

Stepsize constant or changing with time

Tracking a Nonstationary Problem

Choosing Q_k to be a sample average is appropriate in a stationary problem, i.e., when none of the $Q^*(a)$ change over time,

But not in a nonstationary problem.

Better in the nonstationary case is:

$$Q_{k+1} = Q_k + \alpha\left[r_{k+1} - Q_k\right]$$

$$\text{for } constant\ \alpha,\ 0 < \alpha \leq 1$$

$$= (1-\alpha)^k Q_0 + \sum_{i=1}^{k} \alpha(1-\alpha)^{k-i} r_i$$

exponential, recency-weighted average

Computation

Use

$$Q_{k+1} = Q_k + \alpha[r_{k+1} - Q_k]$$

Then

$$\begin{aligned} Q_k &= Q_{k-1} + \alpha[r_k - Q_{k-1}] \\ &= \alpha r_k + (1-\alpha)Q_{k-1} \\ &= \alpha r_k + (1-\alpha)\alpha r_{k-1} + (1-\alpha)^2 Q_{k-2} \\ &= (1-\alpha)^k Q_0 + \sum_{i=1}^{k} \alpha(1-\alpha)^{k-i} r_i \end{aligned}$$

In general : convergence if

$$\sum_{k=1}^{\infty} \alpha_k(a) = \infty \text{ and } \sum_{k=1}^{\infty} \alpha_k^2(a) < \infty$$

satisfied for $\alpha_k = \frac{1}{k}$ but not for fixed α

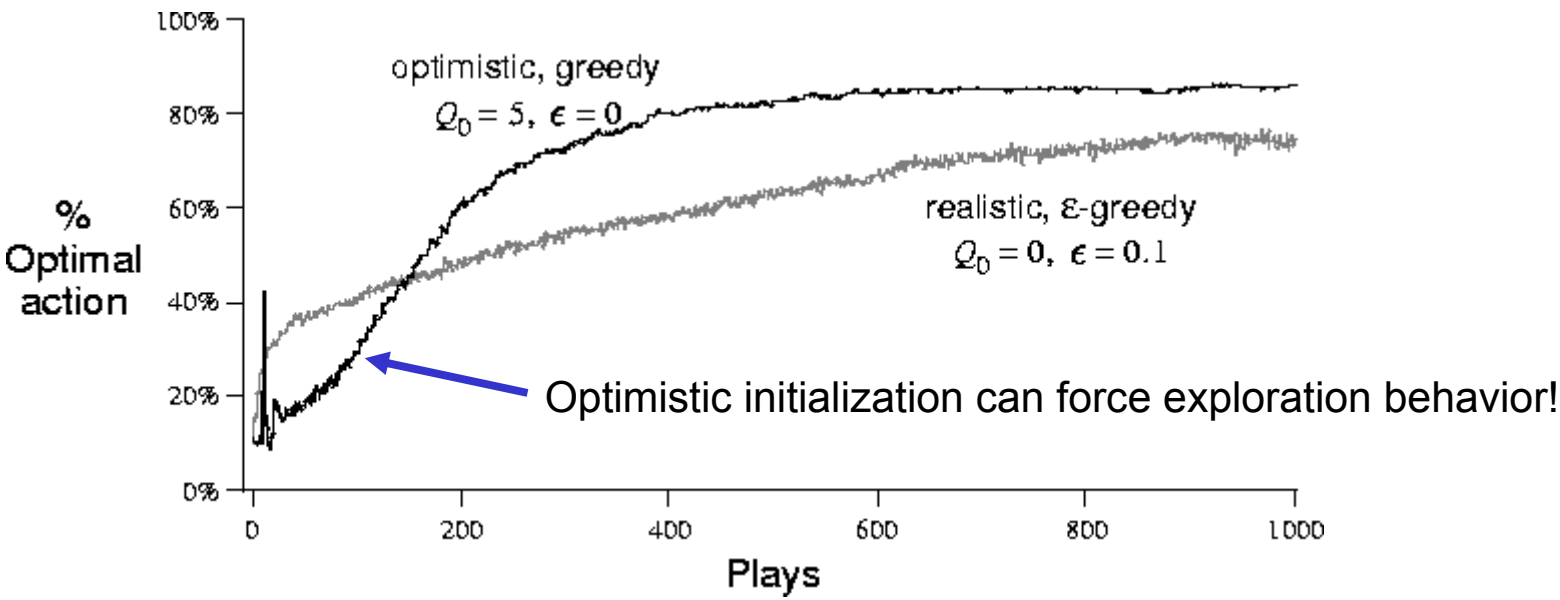
Notes:

1. $\sum_{i=1}^{k} \alpha(1-\alpha)^{k-i} = 1$

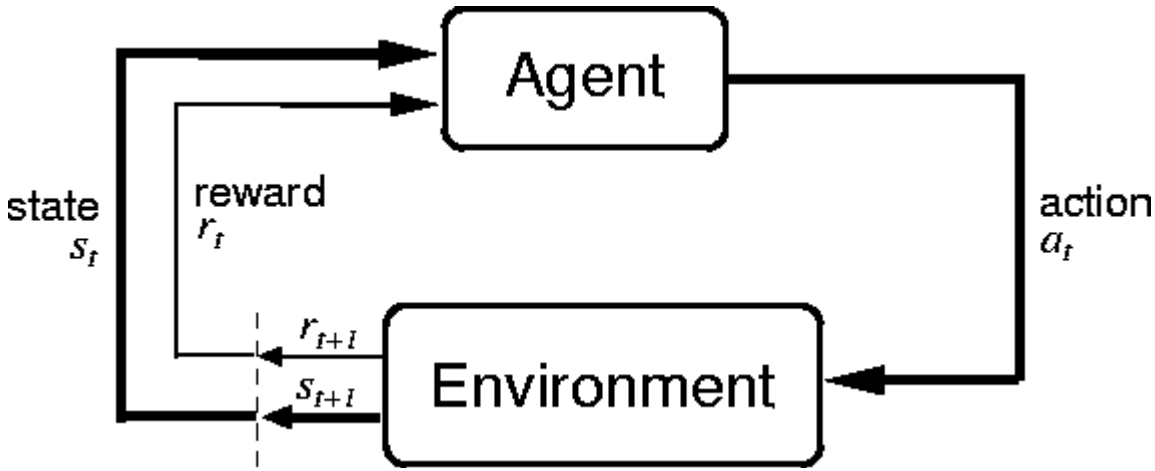
2. Step size parameter after the k-th application of action a

Optimistic Initial Values

- All methods so far depend on $Q_0(a)$, i.e., they are **biased**.
- Suppose instead we initialize the action values **optimistically**, i.e., on the 10-armed testbed, use $Q_0(a) = 5$ for all a.



The Agent-Environment Interface



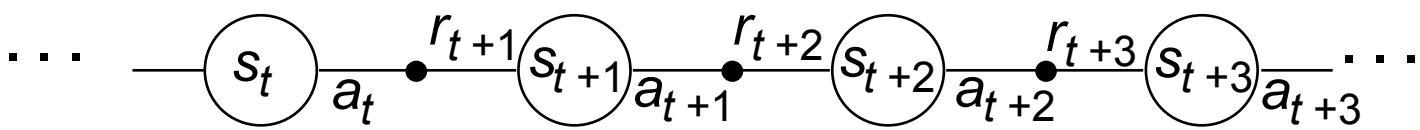
Agent and environment interact at discrete time steps : $t = 0,1,2,\ldots$

Agent observes state at step t : $s_t \in S$

produces action at step t : $a_t \in A(s_t)$

gets resulting reward : $r_{t+1} \in \Re$

and resulting next state : s_{t+1}



The Agent Learns a Policy

Policy at $step\ t, \pi_t$:

$$a\ mapping\ \ from\ states\ to\ action\ probabilities$$

$$\pi_t(s,a) = probability\ that\ a_t = a\ when\ s_t = s$$

- Reinforcement learning methods specify how the agent changes its policy as a result of experience.
- Roughly, the agent's goal is to get as much reward as it can over the long run.

Getting the Degree of Abstraction Right

- Time steps need not refer to fixed intervals of real time.
- Actions can be low level (e.g., voltages to motors), or high level (e.g., accept a job offer), “mental” (e.g., shift in focus of attention), etc.
- States can low-level “sensations”, or they can be abstract, symbolic, based on memory, or subjective (e.g., the state of being “surprised” or “lost”).
- An RL agent is not like a whole animal or robot, which consist of many RL agents as well as other components.
- The environment is not necessarily unknown to the agent, only incompletely controllable.
- Reward computation is in the agent’s environment because the agent cannot change it arbitrarily.

Goals and Rewards

- Is a scalar reward signal an adequate notion of a goal?—maybe not, but it is surprisingly flexible.
- A goal should specify **what** we want to achieve, not **how** we want to achieve it.
- A goal must be outside the agent's direct control—thus outside the agent.
- The agent must be able to measure success:
  - explicitly;
  - frequently during its lifespan.

Returns

Suppose the sequence of rewards after step t is :

$$r_{t+1}, r_{t+2}, r_{t+3}, \cdots$$

What do we want to maximize?

In general,

we want to maximize the **expected return**, $E\{R_t\}$, for each step t.

Episodic tasks: interaction breaks naturally into episodes, e.g., plays of a game, trips through a maze.

$$R_t = r_{t+1} + r_{t+2} + \cdots + r_T,$$

where T is a final time step at which a **terminal state** is reached, ending an episode.

Returns for Continuing Tasks

Continuing tasks: interaction does not have natural episodes.

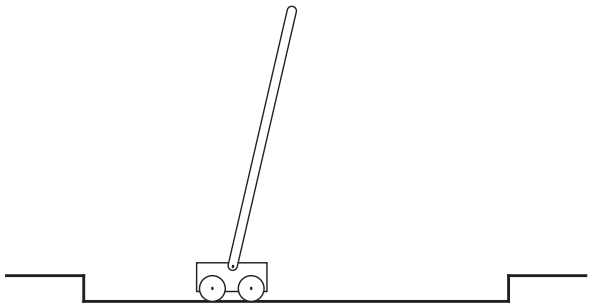
Discounted return:

$$R_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1},$$

where $\gamma, 0 \le \gamma \le 1,$ is the **discount rate**.

$$\text{shortsighted } 0 \leftarrow \gamma \rightarrow 1 \text{ farsighted}$$

An Example



Avoid **failure**: the pole falling beyond a critical angle or the cart hitting end of track.

As an **episodic task** where episode ends upon failure:

$$\text{reward} = +1 \text{ for each step before failure}$$

$$\Rightarrow \text{return} = \text{number of steps before failure}$$

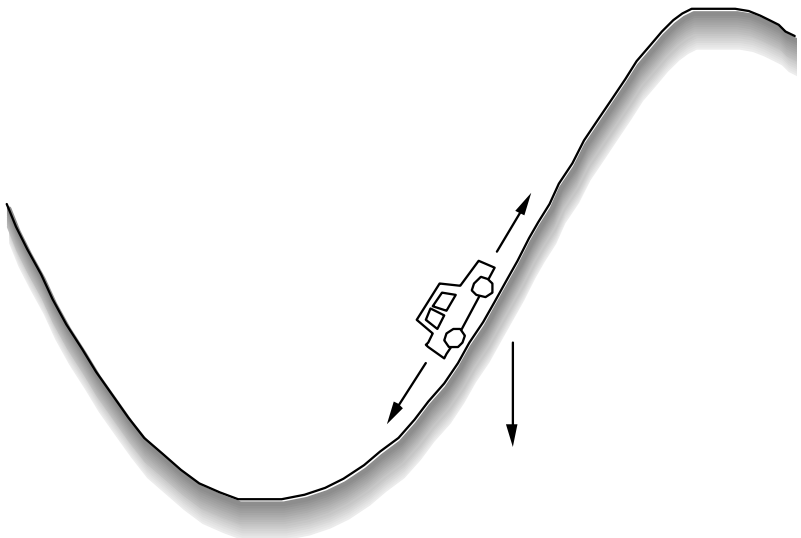
As a **continuing task** with discounted return:

$$\text{reward} = -1 \text{ upon failure; } 0 \text{ otherwise}$$

$$\Rightarrow \text{return} = -\gamma^k, \text{ for } k \text{ steps before failure}$$

In either case, return is maximized by avoiding failure for as long as possible.

Another Example



Get to the top of the hill
as quickly as possible.

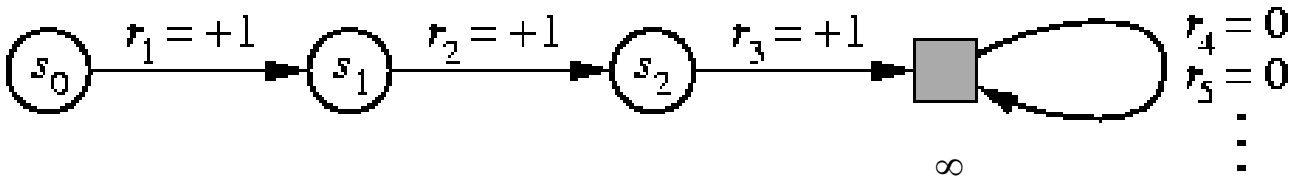
reward $= -1$ for each step where **not** at top of hill

$\Rightarrow$ return $=$ $-$ number of steps before reaching top of hill

Return is maximized by minimizing
number of steps reach the top of the hill.

A Unified Notation

- In episodic tasks, we number the time steps of each episode starting from zero.
- We usually do not have distinguish between episodes, so we write s_t instead of $s_{t,j}$ for the state at step t of episode j.
- Think of each episode as ending in an absorbing state that always produces reward of zero:



- We can cover all cases by writing $R_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$,

  where γ can be 1 only if a zero reward absorbing state is always reached.

The Markov Property

- By "the state" at step *t*, the book means whatever information is available to the agent at step *t* about its environment.
- The state can include immediate "sensations," highly processed sensations, and structures built up over time from sequences of sensations.
- Ideally, a state should summarize past sensations so as to retain all "essential" information, i.e., it should have the **Markov Property**:

$$Pr\{s_{t+1} = s', r_{t+1} = r \mid s_t, a_t, r_t, s_{t-1}, a_{t-1}, \ldots, r_1, s_0, a_0\} =$$
$$Pr\{s_{t+1} = s', r_{t+1} = r \mid s_t, a_t\}$$

for all s', r, and histories s_t, a_t, s_{t-1}, a_{t-1}, ..., r_1, s_0, a_0.

Markov Decision Processes

- If a reinforcement learning task has the Markov Property, it is basically a **Markov Decision Process (MDP)**.
- If state and action sets are finite, it is a **finite MDP**.
- To define a finite MDP, you need to give:
  - **state and action sets**
  - one-step "dynamics" defined by **transition probabilities**:

$$P^a_{ss'} = \Pr\left\{s_{t+1} = s' \,\middle|\, s_t = s, a_t = a\right\} \quad \text{for all } s, s' \in S,\ a \in A(s).$$

  - **reward probabilities**:

$$R^a_{ss'} = E\left\{r_{t+1} \,\middle|\, s_t = s, a_t = a, s_{t+1} = s'\right\} \quad \text{for all } s, s' \in S,\ a \in A(s).$$

An Example Finite MDP

Recycling Robot

- At each step, robot has to decide whether it should (1) actively search for a can, (2) wait for someone to bring it a can, or (3) go to home base and recharge.
- Searching is better but runs down the battery; if runs out of power while searching, has to be rescued (which is bad).
- Decisions made on basis of current energy level: `high`, `low`.
- Reward = number of cans collected

Recycling Robot MDP

$S = \{\text{high}, \text{low}\}$

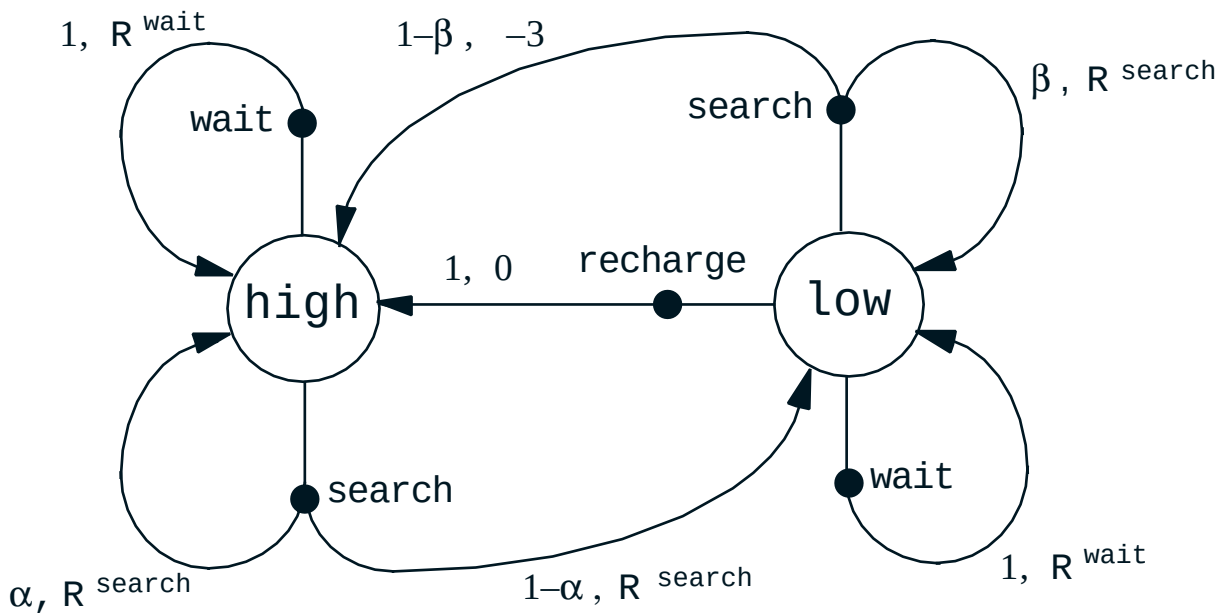
$A(\text{high}) = \{\text{search}, \text{wait}\}$

$A(\text{low}) = \{\text{search}, \text{wait}, \text{recharge}\}$

$R^{\text{search}} = \textit{expected no. of cans while searching}$

$R^{\text{wait}} = \textit{expected no. of cans while waiting}$

$R^{\text{search}} > R^{\text{wait}}$



Transition Table

Table 3.1 Transition probabilities and expected rewards for the finite MDP of the recycling robot example.

s	s'	a	$\mathcal{P}^a_{ss'}$	$\mathcal{R}^a_{ss'}$
high	high	search	α	$\mathcal{R}^{\text{search}}$
high	low	search	$1-\alpha$	$\mathcal{R}^{\text{search}}$
low	high	search	$1-\beta$	-3
low	low	search	β	$\mathcal{R}^{\text{search}}$
high	high	wait	1	$\mathcal{R}^{\text{wait}}$
high	low	wait	0	$\mathcal{R}^{\text{wait}}$
low	high	wait	0	$\mathcal{R}^{\text{wait}}$
low	low	wait	1	$\mathcal{R}^{\text{wait}}$
low	high	recharge	1	0
low	low	recharge	0	0

Note: There is a row for each possible combination of current state, s, next state, s', and action possible in the current state, $a \in \mathcal{A}(s)$.

Value Functions

- The **value of a state** is the expected return starting from that state; depends on the agent's policy:

State-value function for policy π:

$$V^{\pi}(s) = E_{\pi}\{R_t \mid s_t = s\} = E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \mid s_t = s\right\}$$

- The **value of taking an action in a state under policy π** is the expected return starting from that state, taking that action, and thereafter following π:

Action-value function for policy π:

$$Q^{\pi}(s,a) = E_{\pi}\{R_t \mid s_t = s, a_t = a\} = E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \mid s_t = s, a_t = a\right\}$$

Bellman Equation for a Policy π

The basic idea:

$$\begin{aligned} R_t &= r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} \cdots \\ &= r_{t+1} + \gamma \left(r_{t+2} + \gamma r_{t+3} + \gamma^2 r_{t+4} \cdots \right) \\ &= r_{t+1} + \gamma R_{t+1} \end{aligned}$$

So:

$$\begin{aligned} V^\pi(s) &= E_\pi \left\{ R_t \,\middle|\, s_t = s \right\} \\ &= E_\pi \left\{ r_{t+1} + \gamma V\left(s_{t+1}\right) \,\middle|\, s_t = s \right\} \end{aligned}$$

Or, without the expectation operator:

$$V^\pi(s) = \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma V^\pi(s') \right]$$

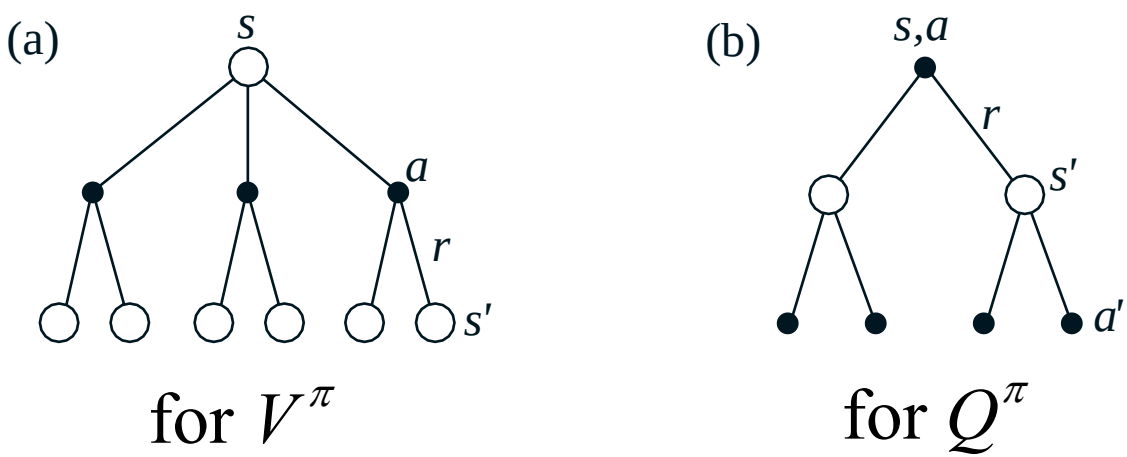
$$
\begin{aligned}
V^{\pi}(s) &= E_{\pi}\{R_t \mid s_t = s\} \\
&= E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s\right\} \\
&= E_{\pi}\left\{r_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k r_{t+k+2} \,\middle|\, s_t = s\right\} \\
&= \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+2} \,\middle|\, s_{t+1} = s'\right\}\right] \\
&= \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V^{\pi}(s')\right], \qquad (3.10)
\end{aligned}
$$

More on the Bellman Equation

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} P^{a}_{ss'} \left[R^{a}_{ss'} + \gamma V^{\pi}(s') \right]$$

This is a set of equations (in fact, linear), one for each state
The value function for π is its unique solution.

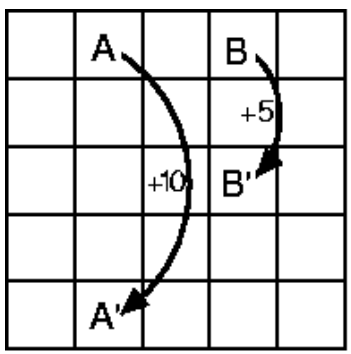
Backup diagrams:



for V^{π} for Q^{π}

Gridworld

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.



(a)

Actions

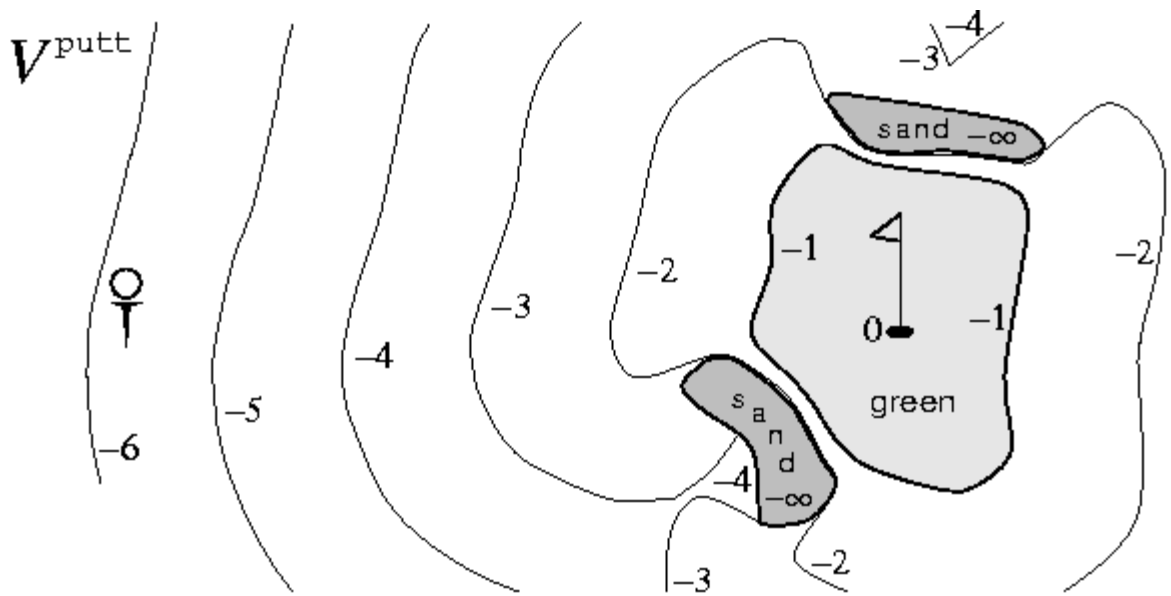
3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

State-value function for equiprobable random policy; $\gamma = 0.9$

Golf

- State is ball location
- Reward of –1 for each stroke until the ball is in the hole
- Value of a state?
- Actions:
  - `putt` (use putter)
  - `driver` (use driver)
- `putt` succeeds anywhere on the green



Optimal Value Functions

- For finite MDPs, policies can be **partially ordered**:

$$\pi \geq \pi' \quad \text{if and only if} \quad V^{\pi}(s) \geq V^{\pi'}(s) \quad \text{for all } s \in S$$

- There is always at least one (and possibly many) policies that is better than or equal to all the others. This is an **optimal policy**. We denote them all π^*.
- Optimal policies share the same **optimal state-value function**:

$$V^*(s) = \max_{\pi} V^{\pi}(s) \quad \text{for all } s \in S$$

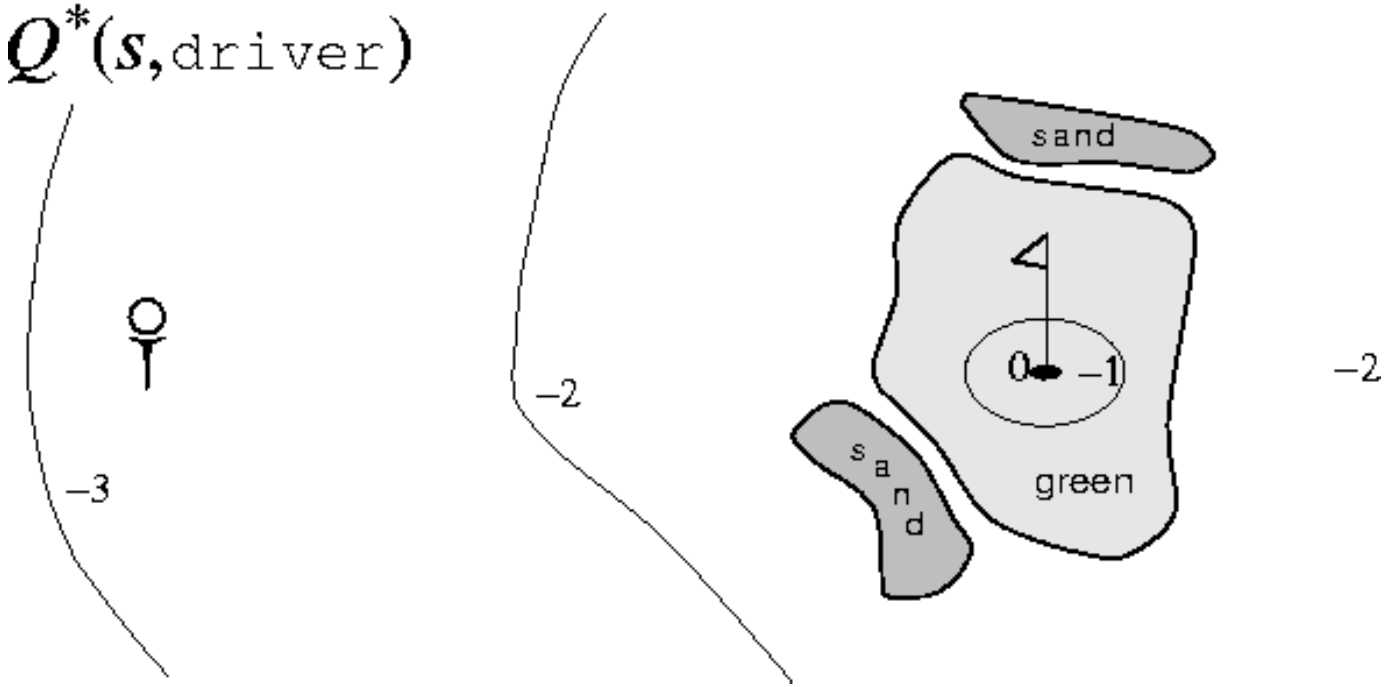
- Optimal policies also share the same **optimal action-value function**:

$$Q^*(s,a) = \max_{\pi} Q^{\pi}(s,a) \quad \text{for all } s \in S \text{ and } a \in A(s)$$

This is the expected return for taking action *a* in state *s* and thereafter following an optimal policy.

Optimal Value Function for Golf

- We can hit the ball farther with `driver` than with `putter`, but with less accuracy
- $Q^*(s,$`driver`$)$ gives the value or using `driver` first, then using whichever actions are best

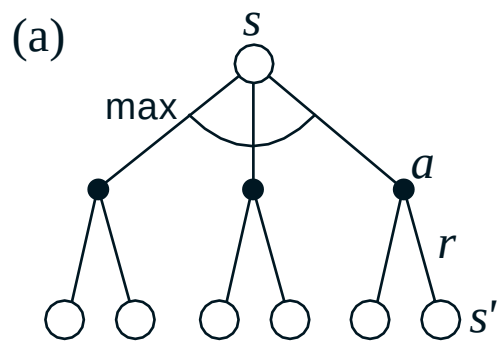


Bellman Optimality Equation for V^*

The value of a state under an optimal policy must equal the expected return for the best action from that state:

$$
\begin{aligned}
V^*(s) &= \max_{a \in A(s)} Q^{\pi^*}(s,a) \\
&= \max_{a \in A(s)} E\left\{ r_{t+1} + \gamma V^*(s_{t+1}) \,\middle|\, s_t = s, a_t = a \right\} \\
&= \max_{a \in A(s)} \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma V^*(s') \right]
\end{aligned}
$$

The relevant backup diagram:

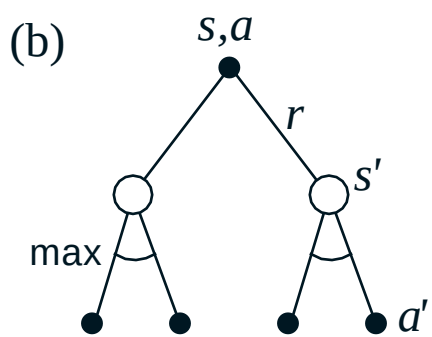


V^* is the unique solution of this system of nonlinear equations.

Bellman Optimality Equation for Q^*

$$Q^*(s,a) = E\left\{ r_{t+1} + \gamma \max_{a'} Q^*(s_{t+1}, a') \,\middle|\, s_t = s, a_t = a \right\}$$
$$= \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma \max_{a'} Q^*(s', a') \right]$$

The relevant backup diagram:



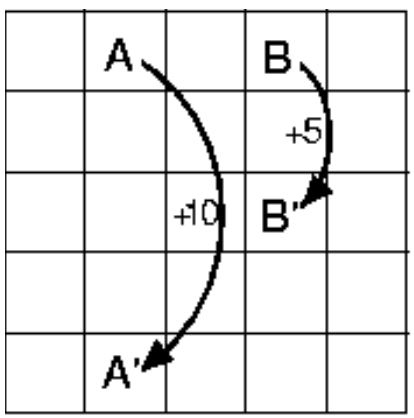
Q^* is the unique solution of this system of nonlinear equations.

Why Optimal State-Value Functions are Useful

Any policy that is greedy with respect to V^* is an optimal policy.

Therefore, given V^*, one-step-ahead search produces the long-term optimal actions.

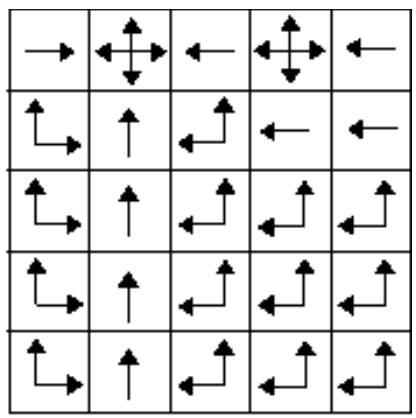
E.g., back to the gridworld:



a) gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

b) V^*



c) π^*

What About Optimal Action-Value Functions?

Given Q^* , the agent does not even have to do a one-step-ahead search:

$$\pi^*(s) = \arg\max_{a \in A(s)} Q^*(s,a)$$

Solving the Bellman Optimality Equation

- Finding an optimal policy by solving the Bellman Optimality Equation requires the following:
  - accurate knowledge of environment dynamics;
  - we have enough space an time to do the computation;
  - the Markov Property.
- How much space and time do we need?
  - polynomial in number of states (via dynamic programming methods; Chapter 4),
  - BUT, number of states is often huge (e.g., backgammon has about 10**20 states).
- We usually have to settle for approximations.
- Many RL methods can be understood as approximately solving the Bellman Optimality Equation.

Summary

- Agent-environment interaction
  - States
  - Actions
  - Rewards
- Policy: stochastic rule for selecting actions
- Return: the function of future rewards agent tries to maximize
- Episodic and continuing tasks
- Markov Property
- Markov Decision Process
  - Transition probabilities
  - Expected rewards
- Value functions
  - State-value function for a policy
  - Action-value function for a policy
  - Optimal state-value function
  - Optimal action-value function
- Optimal value functions
- Optimal policies
- Bellman Equations
- The need for approximation