

Advanced AI Techniques

I. Bayesian Networks / 3. Structure Learning (1/2)

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1. Checking Probabilistic Independencies

2. Markov Equivalence and DAG patterns

3. PC Algorithm

The Very Last Step

- Assume, we know the whole structure of a bn except a single edge.
- This edge represents a single independence statement.
- Check it and include edge based on outcome of that test.

Exact Check / Example (1/3)

If X and Y are independent, then

$$p(X, Y) = p(X) \cdot p(Y)$$

observed

$Y =$	0	1
$X = 0$	3	6
1	1	2

observed relative frequencies $p(X, Y)$:

$Y =$	0	1
$X = 0$	0.25	0.5
1	0.083	0.167

expected relative frequencies
 $p(X) p(Y)$:

$Y =$	0	1
$X = 0$	0.25	0.5
1	0.083	0.167

Exact Check / Example (2/3)

If X and Y are independent, then

$$p(X, Y) = p(X) \cdot p(Y)$$

observed

$Y =$	0	1
$X = 0$	3000	6000
1	1000	2000

observed relative frequencies $p(X, Y)$:

$Y =$	0	1
$X = 0$	0.25	0.5
1	0.083	0.167

expected relative frequencies
 $p(X) p(Y)$:

$Y =$	0	1
$X = 0$	0.25	0.5
1	0.083	0.167

Exact Check / Example (3/3)

If X and Y are independent, then

$$p(X, Y) = p(X) \cdot p(Y)$$

observed

$Y =$	0	1
$X = 0$	2999	6001
1	1000	2000

observed relative frequencies $p(X, Y)$:

$Y =$	0	1
$X = 0$	0.2499167	0.5000833
1	0.0833333	0.1666667

expected relative frequencies

$p(X) p(Y)$:

$Y =$	0	1
$X = 0$	0.2499375	0.5000625
1	0.0833125	0.1666875

Gamma function (repetition, see I.2)

Definition 1. Gamma function

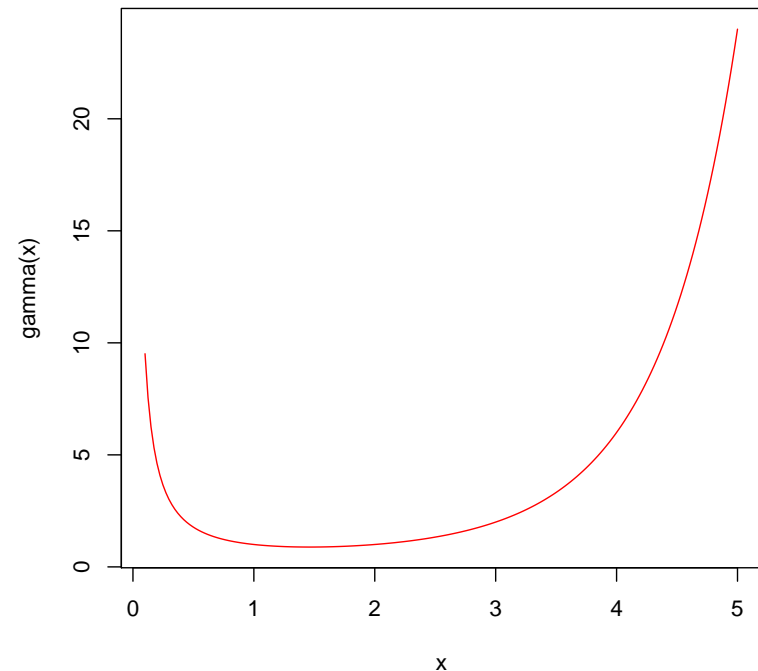
$$\Gamma(a) := \int_0^{\infty} t^{a-1} e^{-t} dt$$

converging for $a > 0$.

Lemma 1 (Γ is generalization of factorial).

(i) $\Gamma(n) = (n - 1)!$ for $n \in \mathbb{N}$.

(ii) $\frac{\Gamma(a+1)}{\Gamma(a)} = a$.



Incomplete Gamma Function

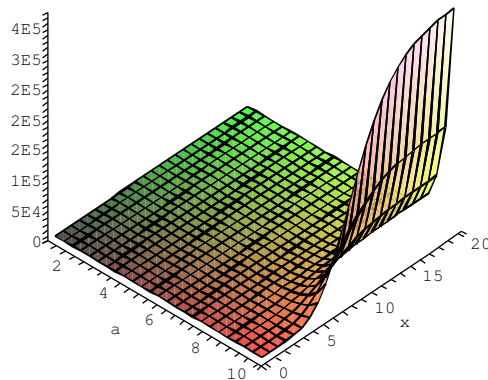
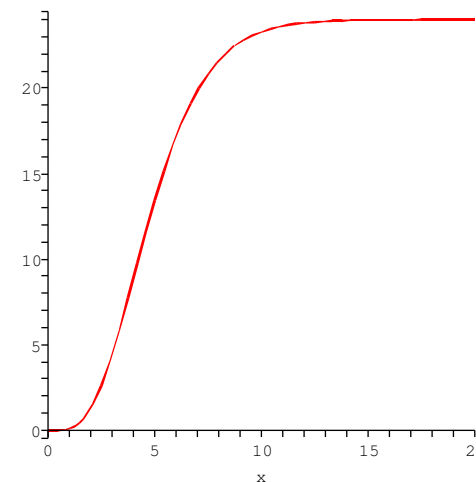
Definition 2. Incomplete Gamma function

$$\gamma(a, x) := \int_0^x t^{a-1} e^{-t} dt$$

defined for $a > 0$ and $x \in [0, \infty]$.

Lemma 2.

$$\gamma(a, \infty) = \Gamma(a)$$

 γ  $\gamma(5, x)$

χ^2 distribution

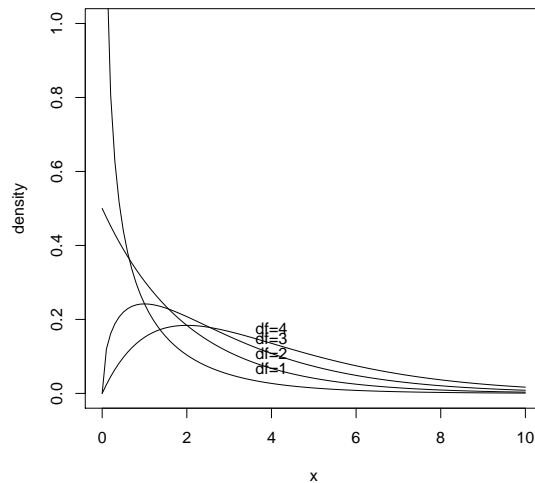
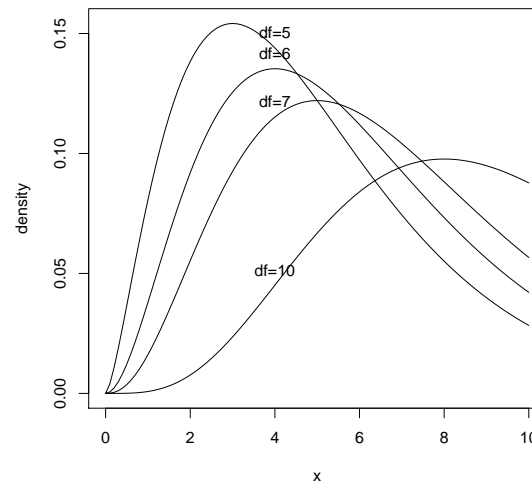
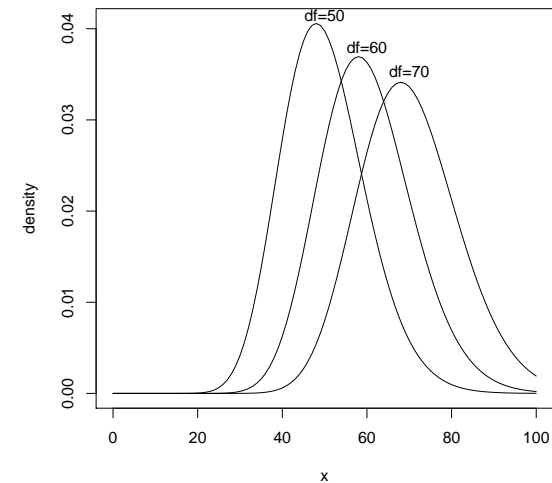
Definition 3. chi-square distribution (χ^2) has density

$$p(x) := \frac{1}{2^{\frac{\text{df}}{2}} \Gamma(\frac{\text{df}}{2})} x^{\frac{\text{df}}{2}-1} e^{-\frac{x}{2}};$$

defined on $]0, \infty[$.

Its cumulative distribution function (cdf) is:

$$p(X < x) := \frac{\gamma(\frac{\text{df}}{2}, \frac{x}{2})}{\Gamma(\frac{\text{df}}{2})};$$


 $\chi_1^2, \chi_2^2, \chi_3^2, \chi_4^2$

 $\chi_5^2, \chi_6^2, \chi_7^2, \chi_{10}^2$

 $\chi_{50}^2, \chi_{60}^2, \chi_{70}^2$

χ^2 distribution**Lemma 3.**

$$E(\chi^2(x, \text{df})) = \text{df}$$

A Java implementation of the incomplete gamma function (and thus of χ^2 distribution) can be found, e.g., in COLT (<http://dsd.lbl.gov/~hoschek/colt/>), package `cern.jet.stat`, **class** `Gamma`.

Be careful, sometimes

$$\tilde{\gamma}(a, x) := \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt = \frac{1}{\Gamma(a)} \gamma(a, x) \quad (\text{e.g., R})$$

or

$$\tilde{\gamma}(a, x) := \int_x^\infty t^{a-1} e^{-t} dt = \Gamma(a) - \gamma(a, x) \quad (\text{e.g., Maple})$$

are referenced as incomplete gamma function.

Count Variables / Just 2 Variables

Let X, Y be random variables, $D \subseteq \text{dom}(X) \times \text{dom}(Y)$
and for two values $x \in \text{dom}(X), y \in \text{dom}(Y)$

$$c_{X=x} := |\{d \in D \mid d|_X = x\}|$$

$$c_{Y=y} := |\{d \in D \mid d|_Y = y\}|$$

$$c_{X=x, Y=y} := |\{d \in D \mid d|_X = x, d|_Y = y\}|$$

their counts.

χ^2 and G^2 statistics / Just 2 Variables

If X, Y are independent, then

$$p(X, Y) = p(X) p(Y)$$

and thus

$$E(c_{X=x, Y=y} \mid c_{X=x}, c_{Y=y}) = \frac{c_{X=x} \cdot c_{Y=y}}{|D|}$$

Then the statistics

$$\chi^2 := \sum_{x \in \text{dom}(X)} \sum_{y \in \text{dom}(Y)} \frac{\left(c_{X=x, Y=y} - \frac{c_{X=x} \cdot c_{Y=y}}{|D|} \right)^2}{\frac{c_{X=x} \cdot c_{Y=y}}{|D|}}$$

as well as

$$G^2 := 2 \sum_{x \in \text{dom}(X)} \sum_{y \in \text{dom}(Y)} c_{X=x, Y=y} \cdot \ln \left(\frac{c_{X=x, Y=y}}{\left(\frac{c_{X=x} \cdot c_{Y=y}}{|D|} \right)} \right)$$

are asymptotically χ^2 -distributed with

$\text{df} = (|\text{dom}(X)| - 1)(|\text{dom}(Y)| - 1)$ degrees of freedom.

Testing Independency / informal

Generally, the statistics have the form

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected})^2}{\text{expected}}$$

$$G^2 = \sum \text{observed} \ln \left(\frac{\text{observed}}{\text{expected}} \right)$$

$\chi^2 = 0$ and $G^2 = 0$ for exact independent variables.

The larger χ^2 and G^2 , the more likely / stronger the dependency between X and Y .

Testing Independency / more formally

More formally, under the

null hypothesis of independency of X and Y ,

the probability for χ^2 and G^2 to have the computed values (or even larger ones) is

$$p_{\chi^2_{df}}(X > \chi^2) \quad \text{and} \quad p_{\chi^2_{df}}(X > G^2)$$

Let p_0 be a given threshold called **significance level** and often chosen as 0.05 or 0.01.

- If $p(X > \chi^2) < p_0$, we can **reject the null hypothesis** and thus accept its

alternative hypothesis of dependency of X and Y .

i.e., add the edge between X and Y .

- If $p(X > \chi^2) \geq p_0$, we cannot reject the null hypothesis. Here, we then will accept the null hypothesis, i.e., not add the edge between X and Y .

Example 1

observed

$Y =$	0	1
$X = 0$	3	6
1	1	2

margins

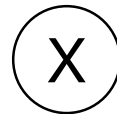
$Y =$	0	1	Σ
$X = 0$	3	6	9
1	1	2	3
Σ	4	8	12

expected

$Y =$	0	1	Σ
$X = 0$	3	6	9
1	1	2	3
Σ	4	8	12

$$\chi^2 = G^2 = 0 \quad \text{and} \quad p(X > 0) = 1$$

Hence, for any significance level
 X and Y are considered independent.



Example 2 (1/2)

observed

$Y =$	0	1
$X = 0$	6	1
1	2	4

margins

$Y =$	0	1	Σ
$X = 0$	6	1	7
1	2	4	6
Σ	8	5	13

expected

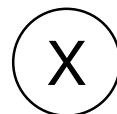
$Y =$	0	1	Σ
$X = 0$	4.31	2.69	7
1	3.69	2.31	6
Σ	8	5	13

$$\chi^2 = \frac{(6 - 4.31)^2}{4.31} + \frac{(1 - 2.69)^2}{2.69} + \frac{(2 - 3.69)^2}{3.69} + \frac{(4 - 2.31)^2}{2.31}$$

$$= 3.75,$$

$$\rightsquigarrow p_{\chi^2_1}(X > 3.75) = 0.053$$

i.e., with a significance level of $p_0 = 0.05$
 we would **not** be able to reject the null hypothesis of
 independency of X and Y .



Example 2 (2/2)

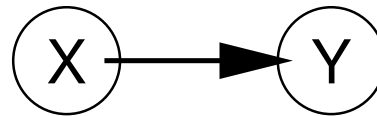
If we use G^2 instead of χ^2 ,

$$G^2 = 3.94, \quad p_{\chi^2_1}(X > 3.94) = 0.047$$

with a significance level of $p_0 = 0.05$

we would have to reject the null hypothesis of independency of X and Y .

Here, we then accept the alternative, dependency of X and Y .



count variables / general case

Let \mathcal{V} be a set of random variables.

We write $v \in \mathcal{V}$ as abbreviation for $v \in \prod \text{dom}(\mathcal{V})$.

For a dataset $D \subseteq \prod \text{dom}(\mathcal{V})$ and

- each subset $\mathcal{X} \subseteq \mathcal{V}$ of variables and
- each configuration $x \in \mathcal{X}$ of these variables

let

$$c_{\mathcal{X}=x} := |\{d \in D \mid d|_{\mathcal{X}} = x\}|$$

be a (random) variable containing the frequencies of occurrences of $\mathcal{X} = x$ in the data.

G^2 statistics / general case

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{V}$ be three disjoint subsets of variables. If

$$I(\mathcal{X}, \mathcal{Y} \mid \mathcal{Z})$$

then

$$p(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \frac{p(\mathcal{X}, \mathcal{Z}) p(\mathcal{Y}, \mathcal{Z})}{p(\mathcal{Z})}$$

and thus for each configuration $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $z \in \mathcal{Z}$

$$E(c_{\mathcal{X}=x, \mathcal{Y}=y, \mathcal{Z}=z} \mid c_{\mathcal{X}=x, \mathcal{Z}=z}, c_{\mathcal{Y}=y, \mathcal{Z}=z}) = \frac{c_{\mathcal{X}=x, \mathcal{Z}=z} c_{\mathcal{Y}=y, \mathcal{Z}=z}}{c_{\mathcal{Z}=z}}$$

The statistics

$$G^2 := 2 \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \sum_{z \in \mathcal{Z}} c_{\mathcal{X}=x, \mathcal{Y}=y, \mathcal{Z}=z} \cdot \ln \left(\frac{c_{\mathcal{X}=x, \mathcal{Y}=y, \mathcal{Z}=z} \cdot c_{\mathcal{Z}=z}}{c_{\mathcal{X}=x, \mathcal{Z}=z} \cdot c_{\mathcal{Y}=y, \mathcal{Z}=z}} \right)$$

is asymptotically χ^2 -distributed with

$$\text{df} = \prod_{X \in \mathcal{X}} (|\text{dom } X| - 1) \prod_{Y \in \mathcal{Y}} (|\text{dom } Y| - 1) \prod_{Z \in \mathcal{Z}} |\text{dom } Z|$$

degrees of freedom.

Recommendations

Recommendations [SGS00, p. 95]:

- As heuristics, reduce degrees of freedom by 1 for each **structural zero**:

$$\text{df}^{\text{reduced}} := \text{df} - |\{(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \mid c_{\mathcal{X}=x, \mathcal{Y}=y, \mathcal{Z}=z} = 0\}|$$

- Use G^2 instead of χ^2 .
- If $|D| < 10 \text{ df}$, assume conditional dependency.

Problems:

- null hypothesis is accepted if it is not rejected.
(especially problematic for small samples)
- repeated testing.

1. Checking Probabilistic Independencies

2. Markov Equivalence and DAG patterns

3. PC Algorithm

Markov-equivalence

Definition 4. Let G, H be two graphs on a set V (undirected or DAGs).

G and H are called **markov-equivalent**, if they have the same independency model, i.e.

$$I_G(X, Y|Z) \Leftrightarrow I_H(X, Y|Z), \quad \forall X, Y, Z \subseteq V$$

The notion of markov-equivalence for undirected graphs is uninteresting, as every undirected graph is markov-equivalent only to itself (corollary of uniqueness of minimal representation!).

Markov-equivalence

Why is markov-equivalence important?

1. in structure learning, the set of all graphs over V is our search space.
 \rightsquigarrow if we can restrict searching to equivalence classes, the search space becomes smaller.
2. if we interpret the edges of our graph as causal relationships between variables, it is of interest,
 - which edges are necessary
 (i.e., occur in all instances of the equivalence class), and
 - which edges are only possible
 (i.e., occur in some instances of the equivalence class, but not in some others; i.e., there are alternative explanations).

Markov-equivalence

Definition 5. Let G be a directed graph. We call a chain

$$p_1 - p_2 - p_3$$

uncoupled if there is no edge between p_1 and p_3 .

Lemma 4 (markov-equivalence criterion, [PGV90]). *Let G and H be two DAGs on the vertices V .*

G and H are markov-equivalent if and only if

(i) G and H have the same links ($u(G) = u(H)$) and

(ii) G and H have the same uncoupled head-to-head meetings.

The set of uncoupled head-to-head meetings is also denoted as **V-structure** of G .

The figure consists of two directed graphs side-by-side. Both graphs have seven nodes: A, B, C, D, E, F, and G. In both graphs, nodes A, C, D, F, and G are white with black outlines, while nodes B and E are orange with black outlines. In the left graph, the edges are: A → B, C → B, C → D, B → E, D → E, E → F, and E → G. In the right graph, which represents the network after node B has been deleted, the edges are: C → C (a self-loop), C → D, D → E, E → F, and E → G. The edge from A to B and the edge from B to E are absent.

(a) $A \rightarrow B \rightarrow C$

(b) $A \rightarrow C \rightarrow B \rightarrow A$

(c) $C \rightarrow B \rightarrow A$

(d) $C \rightarrow B \rightarrow A \rightarrow C$

(e) $C \rightarrow B \rightarrow A$

Wolfram Burgard, Luc de Raedt, Bernhard Nebel, Lars Schmidt-Thieme, Institute for Computer Science, University of Freiburg, Germany,
Course on Advanced AI Techniques, winter term 2004

Directed graph patterns

Definition 6. Let V be a set and $E \subseteq V^2 \cup \mathcal{P}^2(V)$ a set of ordered and unordered pairs of elements of V with $(v, w), (w, v) \notin E$ for $v, w \in V$ with $\{v, w\} \in E$.

Then $G := (V, E)$ is called a **directed graph pattern**. The elements of V are called vertices, the elements of E **edges**: unordered pairs are called **undirected edges**, ordered pairs **directed edges**.

We say, a directed graph pattern H is a **pattern of the directed graph** G , if there is an orientation of the unoriented edges of H that yields G , i.e.

$$(v, w) \in E_G \Rightarrow \begin{cases} (v, w) \in E_H \text{ or} \\ \{v, w\} \in E_H \end{cases}$$

$$(v, w) \in E_G \Leftarrow (v, w) \in E_H$$

$$\left. \begin{array}{l} (v, w) \in E_G \text{ or} \\ (w, v) \in E_G \end{array} \right\} \Leftarrow \{v, w\} \in E_H$$

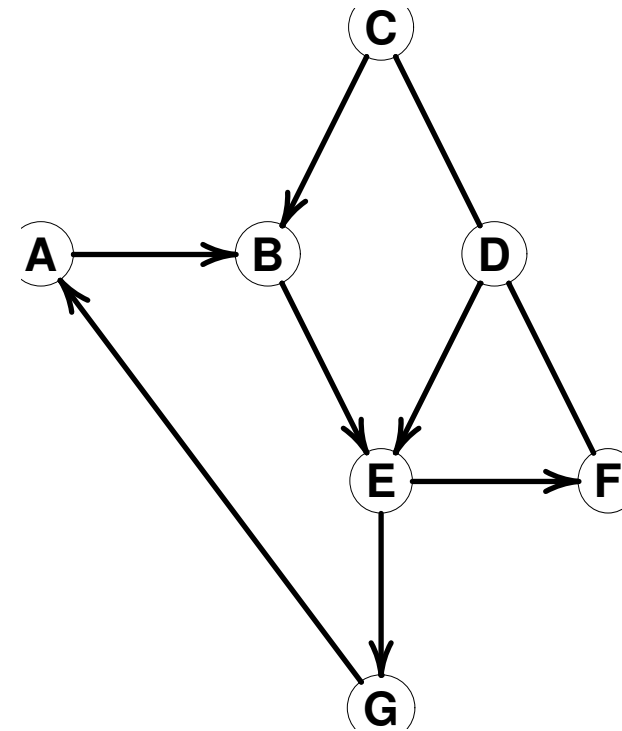


Figure 3: Directed graph pattern.

DAG patterns

Definition 7. A directed graph pattern H is called an **acyclic directed graph pattern** (DAG pattern), if

- it is the directed graph pattern of a DAG G
or equivalently
- G does not contain a completely directed cycle, i.e. there is no sequence $v_1, \dots, v_n \in V$ with $(v_i, v_{i+1}) \in E$ for $i = 1, \dots, n - 1$ (i.e. the directed graph got by dropping undirected edges is a DAG).

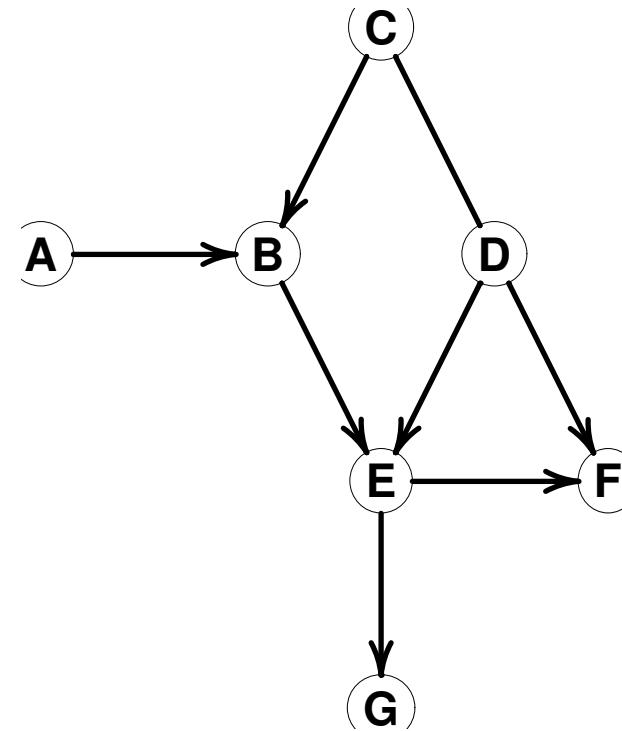


Figure 4: DAG pattern.

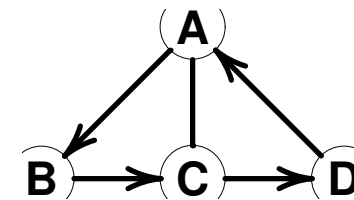


Figure 5: Directed graph pattern that is not a DAG pattern.

DAG patterns represent markov equivalence classes

Lemma 5. *Each markov equivalence class corresponds uniquely to a DAG pattern G :*

The markov equivalence class consists of all DAGs that G is a pattern of, i.e., that give G by dropping the directions of some edges that are not part of an uncoupled head-to-head meeting,

((i) The DAG pattern contains a directed edge (v, w) , if all representatives of the markov equivalence class contain this directed edge, otherwise (i.e. if some representatives have (v, w) , some others (w, v)) the DAG pattern contains the undirected edge $\{v, w\}$.

The directed edges of the DAG pattern are also called **irreversible** or **compelled**, the undirected edges are also called **reversible**.

DAG patterns represent markov equivalence classes / example

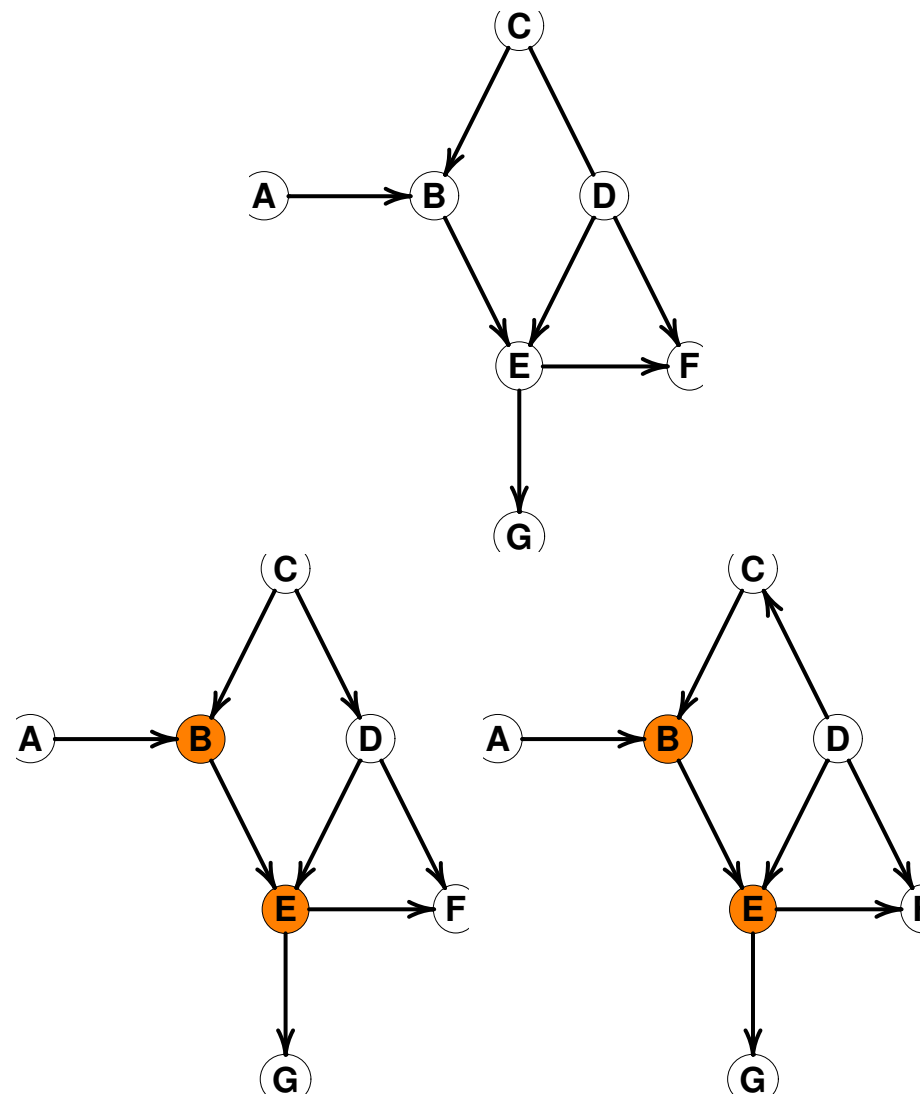
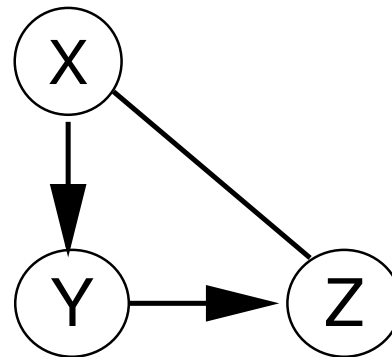


Figure 6: DAG pattern and its markov equivalence class representatives.

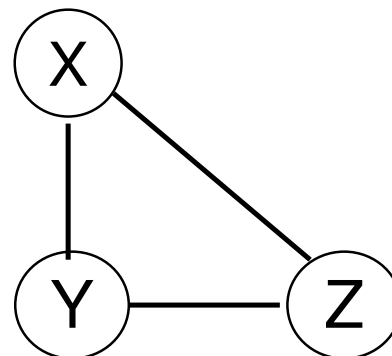
DAG patterns represent markov equivalence classes

But beware, not every DAG pattern represents a Markov-equivalence class !

Example:



is not a DAG pattern of a Markov-equivalence class, but

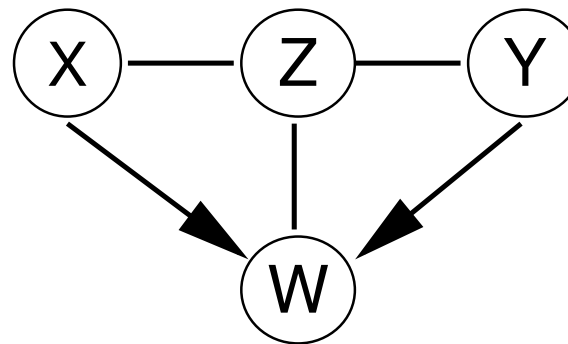


is.

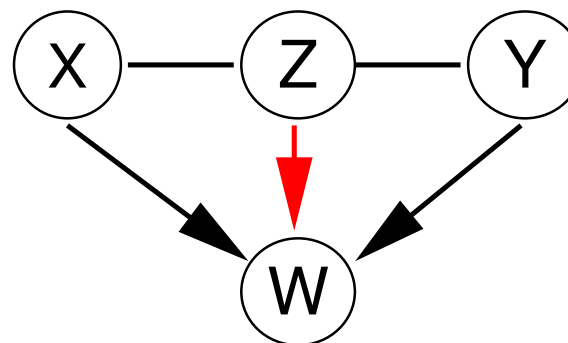
DAG patterns represent markov equivalence classes

But just skeleton plus uncoupled head-to-head meetings do not make a DAG pattern that represents a markov-equivalence class either.

Example:



is not a DAG pattern that represents a Markov-equivalence class, as any of its representatives also has $Z \rightarrow W$. But



is.

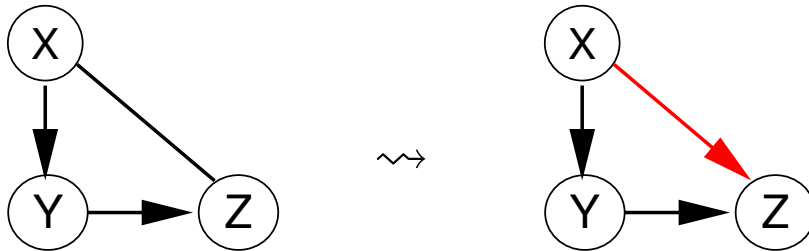
Computing DAG patterns

So, to compute the DAG pattern that represents the equivalence class of a given DAG,

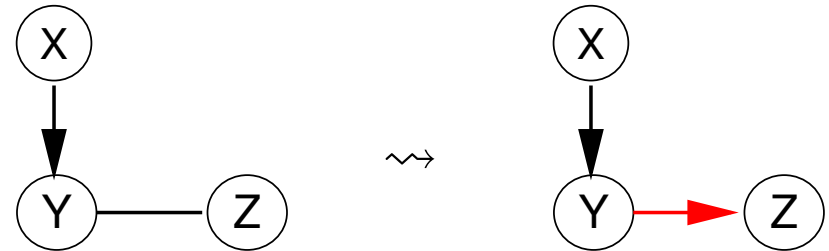
1. start with the skeleton plus all head-to-head-meetings,
2. add entailed edges successively (saturating).

Saturating DAG patterns

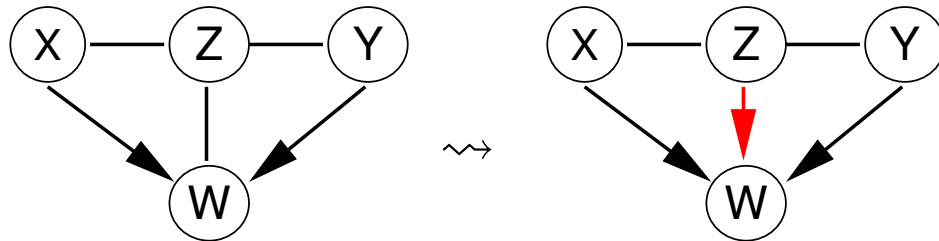
rule 1:



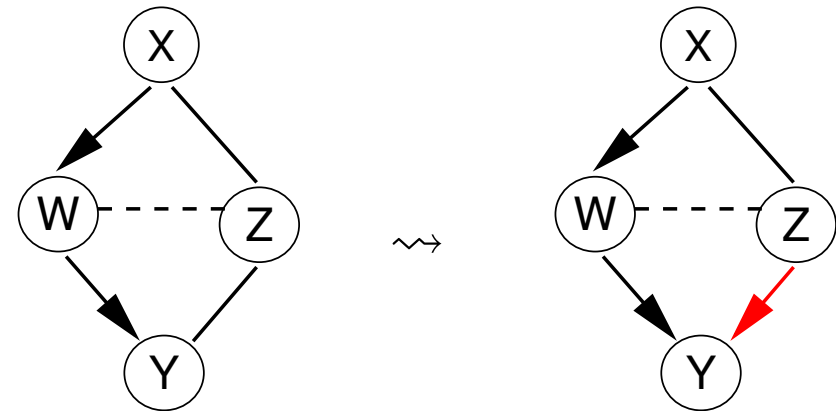
rule 2:



rule 3:



rule 4:



Dashed link can be $W \rightarrow Z$, $W \leftarrow Z$, or $W-Z$
(so rule 4 is actually a compact notation for 3 rules).

Computing DAG patterns

```
1 saturate(graph pattern  $G = (V, E)$ ) :  
2 apply rules 1–4 to  $G$  until no more rule matches  
3 return  $G$   
  
1 dag-pattern(graph  $G = (V, E)$ ) :  
2  $H := (V, F)$  with  $F := \{\{x, y\} \mid (x, y) \in E\}$   
3 for  $X \rightarrow Z \leftarrow Y$  uncoupled head-to-head-meeting in  $G$  do  
4   orient  $X \rightarrow Z \leftarrow Y$  in  $H$   
5 od  
6  $\text{saturate}(H)$   
7 return  $H$ 
```

Figure 7: Algorithm for computing the DAG pattern of the Markov-equivalence class of a given DAG.

Lemma 6. *For a given graph G , algorithm 7 computes correctly the DAG pattern that represents its Markov-equivalence class. Furthermore, here, even the rule set 1–3 will do and is non-redundant.*

See [Mee95] for a proof.

Computing DAG patterns

What follows, is an alternative algorithm for computing DAG patterns that represent the Markov-equivalence class of a given DAG.

Topological edge ordering

Definition 8. Let $G := (V, E)$ be a directed graph.

A bijective map

$$\tau : \{1, \dots, |E|\} \rightarrow E$$

is called an **ordering of the edges of G** .

An edge ordering τ is called **topological edge ordering** if

(i) numbers increase on all paths, i.e.

$$\tau^{-1}(x, y) < \tau^{-1}(y, z)$$

for paths $x \rightarrow y \rightarrow z$ and

(ii) shortcuts have larger numbers, i.e. for x, y, z with

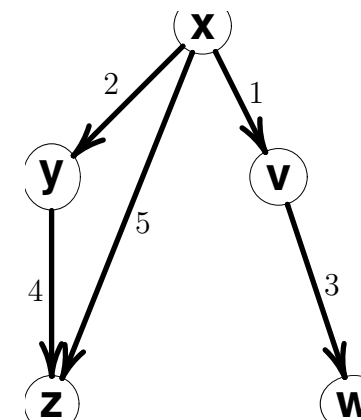
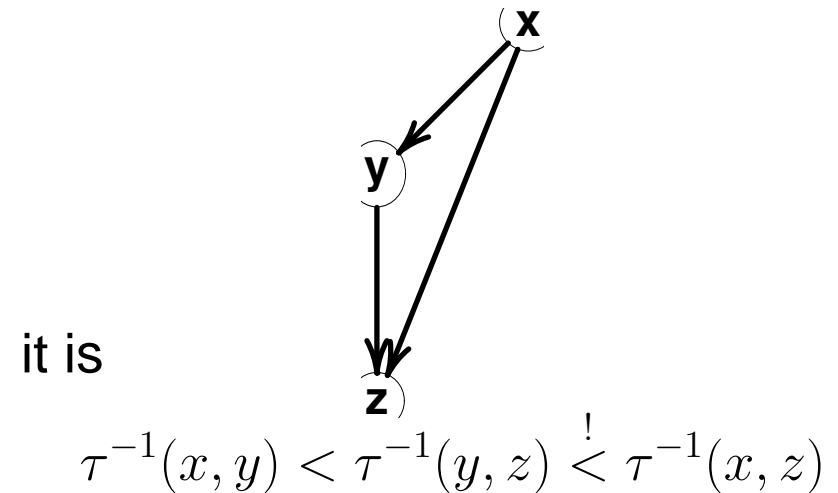


Figure 8: Example for a topological edge ordering.

Topological edge ordering

```
1 topological-edge-ordering( $G = (V, E)$ ) :  
2  $\sigma := \text{topological-ordering}(G)$   
3  $E' := E$   
4 for  $i = 1, \dots, |E|$  do  
5   Let  $(v, w) \in E'$  with  $\sigma^{-1}(w)$  minimal and then with  $\sigma^{-1}(v)$  maximal  
6    $\tau(i) := (v, w)$   
7    $E' := E' \setminus \{(v, w)\}$   
8 od  
9 return  $\tau$ 
```

Figure 9: Algorithm for computing a topological edge ordering of a DAG.

```

1 dag-pattern( $G = (V, E)$ ) :
2  $\tau := \text{topological-edge-ordering}(G)$ 
3  $E_{\text{irr}} := \emptyset$ 
4  $E_{\text{rev}} := \emptyset$ 
5  $E_{\text{rest}} := E$ 
6 while  $E_{\text{rest}} \neq \emptyset$  do
7   Let  $(y, z) \in E_{\text{rest}}$  with  $\tau^{-1}(y, z)$  minimal
8   [label  $\text{pa}(z) :$ ]
9   if  $\exists (x, y) \in E_{\text{irr}}$  with  $(x, z) \notin E$ 
10     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid x' \in \text{pa}(z)\}$ 
11  else
12     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', y) \in E_{\text{irr}}\}$ 
13    if  $\exists (x, z) \in E$  with  $x \notin \{y\} \cup \text{pa}(y)$ 
14       $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
15    else
16       $E_{\text{rev}} := E_{\text{rev}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
17    fi
18  fi
19   $E_{\text{rest}} := E \setminus E_{\text{irr}} \setminus E_{\text{rev}}$ 
20 od
21 return  $\bar{G} := (V, E_{\text{irr}} \cup \{\{v, w\} \mid (v, w) \in E_{\text{rev}}\})$ 

```

Figure 10: Algorithm for computing the DAG pattern representing the markov equivalence class of a DAG G . [Chi95]

A simple but important lemma

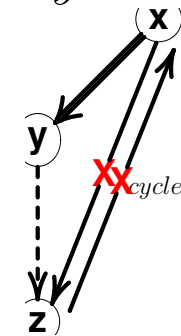
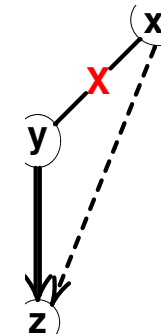
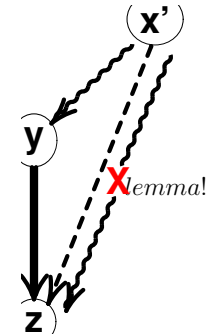
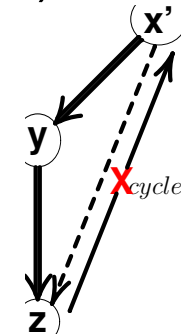
Lemma 7 ([Chi95]). *Let G be a DAG and x, y, z three vertices of G that are pairwise adjacent.
If any two of the connecting edges are reversible, then the third one is also.*

line 10

```

1 dag-pattern( $G = (V, E)$ ) :
2  $\tau := \text{topological-edge-ordering}(G)$ 
3  $E_{\text{irr}} := \emptyset$ 
4  $E_{\text{rev}} := \emptyset$ 
5  $E_{\text{rest}} := E$ 
6 while  $E_{\text{rest}} \neq \emptyset$  do
7   Let  $(y, z) \in E_{\text{rest}}$  with  $\tau^{-1}(y, z)$  minimal
8   [label  $\text{pa}(z)$  :]
9   if  $\exists(x, y) \in E_{\text{irr}}$  with  $(x, z) \notin E$ 
10     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid x' \in \text{pa}(z)\}$ 
11  else
12     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', y) \in E_{\text{irr}}\}$ 
13    if  $\exists(x, z) \in E$  with  $x \notin \{y\} \cup \text{pa}(y)$ 
14       $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
15    else
16       $E_{\text{rev}} := E_{\text{rev}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
17    fi
18  fi
19   $E_{\text{rest}} := E \setminus E_{\text{irr}} \setminus E_{\text{rev}}$ 
20 od
21 return  $\bar{G} := (V, E_{\text{irr}} \cup \{(v, w) \mid (v, w) \in E_{\text{rev}}\})$ 

```

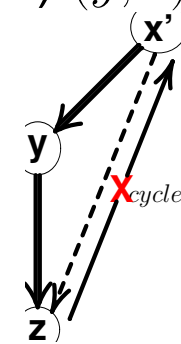
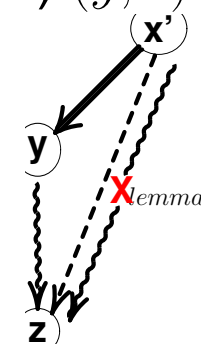
a) $x' = y$:b) $x' \neq y$: case 1) x' and y not adjacent:case 2) x' and y adjacent:

line 12

```

1 dag-pattern( $G = (V, E)$ ) :
2  $\tau := \text{topological-edge-ordering}(G)$ 
3  $E_{\text{irr}} := \emptyset$ 
4  $E_{\text{rev}} := \emptyset$ 
5  $E_{\text{rest}} := E$ 
6 while  $E_{\text{rest}} \neq \emptyset$  do
7   Let  $(y, z) \in E_{\text{rest}}$  with  $\tau^{-1}(y, z)$  minimal
8   [label pa( $z$ ) :]
9   if  $\exists (x, y) \in E_{\text{irr}}$  with  $(x, z) \notin E$ 
10     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid x' \in \text{pa}(z)\}$ 
11  else
12     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', y) \in E_{\text{irr}}\}$ 
13    if  $\exists (x, z) \in E$  with  $x \notin \{y\} \cup \text{pa}(y)$ 
14       $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
15    else
16       $E_{\text{rev}} := E_{\text{rev}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
17    fi
18  fi
19   $E_{\text{rest}} := E \setminus E_{\text{irr}} \setminus E_{\text{rev}}$ 
20 od
21 return  $\bar{G} := (V, E_{\text{irr}} \cup \{(v, w) \mid (v, w) \in E_{\text{rev}}\})$ 

```

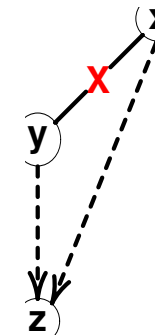
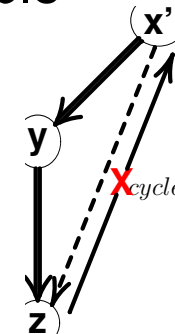
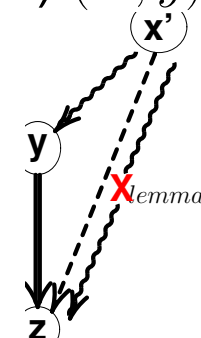
case 1) (y, z) is irreversible:case 2) (y, z) is reversible:

line 14

```

1 dag-pattern( $G = (V, E)$ ) :
2  $\tau := \text{topological-edge-ordering}(G)$ 
3  $E_{\text{irr}} := \emptyset$ 
4  $E_{\text{rev}} := \emptyset$ 
5  $E_{\text{rest}} := E$ 
6 while  $E_{\text{rest}} \neq \emptyset$  do
7   Let  $(y, z) \in E_{\text{rest}}$  with  $\tau^{-1}(y, z)$  minimal
8   [label  $\text{pa}(z)$  :]
9   if  $\exists (x, y) \in E_{\text{irr}}$  with  $(x, z) \notin E$ 
10     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid x' \in \text{pa}(z)\}$ 
11  else
12     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', y) \in E_{\text{irr}}\}$ 
13    if  $\exists (x, z) \in E$  with  $x \notin \{y\} \cup \text{pa}(y)$ 
14     $E_{\text{irr}} := E_{\text{irr}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
15    else
16     $E_{\text{rev}} := E_{\text{rev}} \cup \{(x', z) \mid (x', z) \in E_{\text{rest}}\}$ 
17    fi
18  fi
19   $E_{\text{rest}} := E \setminus E_{\text{irr}} \setminus E_{\text{rev}}$ 
20 od
21 return  $\bar{G} := (V, E_{\text{irr}} \cup \{(v, w) \mid (v, w) \in E_{\text{rev}}\})$ 

```

a) $x' = x$:b) $x' \neq x$: case 1) (x', y) irreversiblecase 2) (x', y) is reversible:

1. Checking Probabilistic Independencies

2. Markov Equivalence and DAG patterns

3. PC Algorithm

...

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